

Coulomb scattering in the massless Nelson model I. Foundations of two-electron scattering

W. Dybalski and A. Pizzo

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Abstract

We construct two-electron scattering states and verify their tensor product structure in the infrared-regular massless Nelson model. The proof follows the lines of Haag-Ruelle scattering theory: Scattering state approximants are defined with the help of two time-dependent renormalized creation operators of the electrons acting on the vacuum. They depend on ground state wave functions of the (single-electron) fiber Hamiltonians with infrared cut-off. Convergence of these approximants as $t \rightarrow \infty$ is shown with the help of Cook's method combined with a non-stationary phase argument. Removal of the infrared cut-off in the limit $t \rightarrow \infty$ requires sharp estimates on the derivatives of these ground state wave functions w.r.t. electron and photon momenta, with mild dependence on the infrared cut-off. These key estimates, which carry information about the localization of electrons in space, are obtained in a companion paper with the help of iterative analytic perturbation theory. Our results hold in the weak coupling regime.

1 Introduction and results

The last two decades have witnessed substantial progress in the mathematical understanding of scattering of light and matter in the framework of non-relativistic QED. These advances provided a rigorous foundation for the physical description of the Compton [CFP07, CFP09, Pi03, Pi05, FGS04] and Rayleigh scattering [DG99, DG04, FGS02], involving one massive particle (the 'electron') and many massless excitations ('photons'). However, the case of Coulomb scattering, i.e., collisions of two electrons in the presence of photons, has remained outside of the scope of these investigations. This is a serious gap in our understanding, given the tremendous importance of this scattering process, ranging from Rutherford's discovery of the structure of atoms, to modern high-energy physics experiments. This paper is a first part of a larger investigation, whose goal is to put this important process on rigorous grounds.

A general framework of scattering theory for several electrons in models of non-relativistic QED, which has its roots in Haag-Ruelle scattering theory [Ha58, Ru62], was known to experts already more than three decades ago [Fr, Al73]. However, a complete construction of scattering states was only possible in the presence of a fixed infrared cut-off (or non-zero photon mass), due to severe infrared- and infraparticle problems. These difficulties were overcome at the single-electron level with the help of a novel multiscale technique [Pi03, Pi05], but the case of several electrons remained open to date. In this paper we provide a construction of

two-electron scattering states in the massless, infrared-regular Nelson model. Although the infraparticle problem does not arise in this situation, the infrared structure of the model is non-trivial and requires major refinements of the multiscale technique. To clarify the origin of these new difficulties and describe our methods to tackle them, let us now explain in non-technical terms the main steps of our analysis.

Let H be the Hamiltonian of the translationally invariant, infrared-regular, massless Nelson model stated in (1.24) below. It describes second-quantized non-relativistic particles, which we call electrons though they obey the Bose statistics, interacting with massless scalar bosons, which we call photons. This Hamiltonian is a self-adjoint operator on the physical Hilbert space \mathcal{H} which is the tensor product of the electron Fock space $\Gamma(\mathfrak{h}_e)$ and the photon Fock space $\Gamma(\mathfrak{h}_f)$. As H preserves the number of electrons, we can restrict it to the one-electron subspace obtaining the Hamiltonian $H^{(1)}$. This Hamiltonian has the standard decomposition into the fiber Hamiltonians at fixed total momentum $H_P^{(1)}$, which are operators on the Fock space. In the infrared-regular case ($1/2 \geq \bar{\alpha} > 0$ in (1.21) below), the fiber Hamiltonians have (normalized) ground states ψ_P , (for P in some ball S centered at zero), corresponding to eigenvalues E_P . Superpositions of such ground states of the form

$$\psi_h := \Pi^* \int^{\oplus} d^3 P h(P) \psi_P, \quad h \in C_0^2(S), \quad (1.1)$$

give physical single-electron states in \mathcal{H} . (Here Π^* is the standard identification between the fiber picture and the physical picture). We note that the time evolution of ψ_h is given by

$$e^{-iHt} \psi_h = \psi_{h_t}, \quad (1.2)$$

where $h_t(P) := e^{-iE_P t} h(P)$. In heuristic terms, to construct a scattering state of two electrons one has to ‘multiply’ two single-electron states in such a way that the result is a vector in \mathcal{H} . Haag-Ruelle scattering theory is, in essence, a prescription to perform such a multiplication. In the context of the Nelson model one could attempt to implement this construction in the following way: Let $\{f_P^m\}_{m \in \mathbb{N}_0}$ be the m -photon components of ψ_P . Then (1.1) can be rewritten as follows

$$\psi_h = \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \int d^3 p d^3 k h(p) f_p^m(k_1, \dots, k_m) a^*(k_1) \dots a^*(k_m) \eta^*(p - \underline{k}) \Omega, \quad (1.3)$$

where $\Omega \in \mathcal{H}$ is the vacuum vector, $\eta^*(p)$, $a^*(k)$ are the electron and photon creation operators and $\underline{k} := k_1 + \dots + k_m$. This suggests the following definition of the *renormalized creation operator*

$$\hat{\eta}^*(h) := \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \int d^3 p d^3 k h(p) f_p^m(k_1, \dots, k_m) a^*(k_1) \dots a^*(k_m) \eta^*(p - \underline{k}), \quad (1.4)$$

which creates the physical single-electron state ψ_h from the vacuum¹. Now, given $h_1, h_2 \in C_0^2(S)$ with disjoint (velocity) supports, the scattering state $\tilde{\Psi}_{h_1, h_2}^+$ describing an asymptotic configuration of two independent electrons ψ_{h_1}, ψ_{h_2} is given by

$$\tilde{\Psi}_{h_1, h_2}^+ = \lim_{t \rightarrow \infty} e^{iHt} \hat{\eta}^*(h_{1,t}) \hat{\eta}^*(h_{2,t}) \Omega, \quad (1.5)$$

¹We found this definition of the renormalized creation operator and a construction of scattering states of several electrons in the Nelson model with a fixed infrared cut-off in unpublished notes of J. Fröhlich [Fr]. Such renormalized creation operators appear also in a work of S. Albeverio [Al73].

if the approximating vectors on the r.h.s. are well defined and the limit exists.

Unfortunately, regularity properties of the m -photon components f_P^m of ψ_P , which control the rate of convergence in (1.5), are difficult to obtain due to the fact that the eigenvalue E_P is located at the bottom of the continuous spectrum of $H_P^{(1)}$. Therefore, we will not construct scattering states with the help of formula (1.5), but instead we will use more tractable approximating sequences: Let $H_{P,\sigma}^{(1)}$ be the fiber Hamiltonians with the infrared cut-off σ , defined precisely in (1.30). Their ground states $\psi_{P,\sigma}$, corresponding to eigenvalues $E_{P,\sigma}$, satisfy

$$\lim_{\sigma \rightarrow 0} \psi_{P,\sigma} = \psi_P. \quad (1.6)$$

Let $\{f_{P,\sigma}^m\}_{m \in \mathbb{N}_0}$ be the m -photon components of $\psi_{P,\sigma}$. The renormalized creation operators with the infrared cut-off σ have the form

$$\hat{\eta}_\sigma^*(h) := \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \int d^3p d^{3m}k h(p) f_{p,\sigma}^m(k_1, \dots, k_m) a^*(k_1) \dots a^*(k_m) \eta^*(p - \underline{k}). \quad (1.7)$$

We will use them to construct the scattering states as follows

$$\Psi_{h_1, h_2}^+ := \lim_{t \rightarrow \infty} e^{iHt} \hat{\eta}_{\sigma_t}^*(h_{1,t}) \hat{\eta}_{\sigma_t}^*(h_{2,t}) \Omega, \quad (1.8)$$

where $\sigma_t = \kappa/t^\gamma$, $\gamma > 4$ and $\kappa > 0$ is the ultraviolet cut-off. The existence of this limit and its properties, which allow to interpret it as an asymptotic configuration of two electrons, constitute the main result of this paper (Theorem 1.3 below). The main steps of the proof, which rely on Cook's method combined with non-stationary phase analysis, are given in Section 2. We conjecture that the limit (1.5) also exists and coincides with (1.8). Supporting evidence for this conjecture comes from algebraic quantum field theory, where the existence of scattering states in the infrared-regular situation has been proven without introducing cut-offs [Dy05]. We do not expect any problems with generalizing our result to an arbitrary number of electrons or with changing the statistics of our 'electrons' from Bose to Fermi. However, our aim here is not to cover the most general situation, but rather to prepare grounds for our future investigation concerning Coulomb scattering in the infrared-singular case. The additional complications coming from the presence of infrared photon clouds should first be tackled in the case of two electrons.

Let us conclude this introductory discussion with some more technical remarks. We point out that the approximating sequence (1.8) has important advantages over (1.5): Due to the fact that $E_{P,\sigma}$ is an isolated eigenvalue, the relevant regularity properties of the eigenvectors $\psi_{P,\sigma}$ and their m -photon components $f_{P,\sigma}^m$ can be obtained using the analytic perturbation theory. Their behaviour in the limit $\sigma \rightarrow 0$ can be studied with the help of the iterative multiscale technique. As this analysis is presented in a separate paper [DP12], it suffices to indicate here the new spectral difficulties encountered at the level of two-electron scattering.

We recall that already in [Fr73, Fr74] a formula for $f_{P,\sigma}^m$ was derived, which in the case of $m = 1$ has the form

$$f_{P,\sigma}^1(k) = -\langle \Omega, \left\{ \frac{1}{H_{P-k,\sigma}^{(1)} - E_{P,\sigma} + |k|} v_{\vec{\alpha}}^\sigma(k) \right\} \psi_{P,\sigma} \rangle, \quad (1.9)$$

where the form-factor $v_{\vec{\alpha}}^\sigma$ is given by (1.31). It is not difficult to obtain from this formula that

$$|f_{P,\sigma}^1(k)| \leq \frac{c v_{\vec{\alpha}}^\sigma(k)}{|k|}, \quad (1.10)$$

where the constant c is independent of σ . However, to prove convergence of (1.8) via the non-stationary phase method, we also need bounds on derivatives $\partial_{P_i} f_{P,\sigma}^m$, $\partial_{P_i} \partial_{P_j} f_{P,\sigma}^m$ with mild dependence on σ . To indicate various difficulties that have to be tackled to obtain such bounds, let us differentiate (1.9) w.r.t. P :

$$\begin{aligned} \partial_{P_i} f_{P,\sigma}^1(k) &= \langle \Omega, \left\{ \frac{1}{H_{P-k,\sigma}^{(1)} - E_{P,\sigma} + |k|} ((P - k - P_f) - \nabla E_{P,\sigma}) \frac{1}{H_{P-k,\sigma}^{(1)} - E_{P,\sigma} + |k|} v_{\vec{\alpha}}^\sigma(k) \right\} \psi_{P,\sigma} \rangle \\ &\quad - \langle \Omega, \left\{ \frac{1}{H_{P-k,\sigma}^{(1)} - E_{P,\sigma} + |k|} v_{\vec{\alpha}}^\sigma(k) \right\} \partial_{P_i} \psi_{P,\sigma} \rangle, \end{aligned} \quad (1.11)$$

where P_f is the photon momentum operator. As for the second term on the r.h.s. of (1.11) the main difficulty is to find an estimate on the vector $\partial_{P_i} \psi_{P,\sigma}$ with suitable dependence on σ . We recall that the existing bounds from [Pi03] give only Hölder continuity of $P \mapsto \psi_{P,\sigma}$, uniformly in σ . By a refinement of the argument from [Pi03], we obtain in [DP12] the following bound, stated precisely in Proposition 1.1 below,

$$\|\partial_{P_i} \psi_{P,\sigma}\| \leq \frac{c}{\sigma^{\delta_{\lambda_0}}}, \quad (1.12)$$

where c is independent of σ , $\lambda_0 > 0$ is the maximal admissible value of the coupling constant and the function $\lambda_0 \mapsto \delta_{\lambda_0}$ satisfies $\lim_{\lambda_0 \rightarrow 0} \delta_{\lambda_0} = 0$. Although the bound is not uniform in σ , such mild dependence on the infrared cut-off is sufficient for our purposes. Denoting by II the second term on the r.h.s. of (1.11), and proceeding as in the derivation of (1.10), we obtain

$$|II| \leq \frac{1}{\sigma^{\delta_{\lambda_0}}} \frac{c v_{\vec{\alpha}}^\sigma(k)}{|k|}. \quad (1.13)$$

As for the first term on the r.h.s. of (1.11), which we denote by I , a crude estimate gives

$$|I| \leq \frac{c v_{\vec{\alpha}}^\sigma(k)}{|k|^2} \leq \frac{1}{\sigma} \frac{c v_{\vec{\alpha}}^\sigma(k)}{|k|}, \quad (1.14)$$

due to the presence of the additional resolvent and the support properties of $v_{\vec{\alpha}}^\sigma$. This bound does not suffice for the purpose of constructing scattering states. With the help of the multiscale analysis we improve it to

$$|I| \leq \frac{1}{\sigma^{\delta_{\lambda_0}}} \frac{c v_{\vec{\alpha}}^\sigma(k)}{|k|}, \quad (1.15)$$

which is only slightly worse than (1.10). Altogether we get

$$|\partial_{P_i} f_{P,\sigma}^1(k)| \leq \frac{1}{\sigma^{\delta_{\lambda_0}}} \frac{c v_{\vec{\alpha}}^\sigma(k)}{|k|} \quad (1.16)$$

and an analogous bound for the second derivative. These bounds, and their counterparts for $f_{P,\sigma}^m$, $m \geq 1$, constitute the main result of [DP12], stated also in Theorem 1.2 below.

This paper is organized as follows: In Subsection 1.1 we recall the definition of the Nelson model with many electrons. In Subsection 1.2 we state the relevant results concerning spectral theory, which are proven in a separate paper [DP12]. In Subsection 1.3 we state the main result of the present paper which concern scattering theory of two electrons in the infrared-regular massless Nelson model. Section 2 presents the main steps of the proof and in the

subsequent sections we provide the necessary ingredients. Section 3 is devoted to vacuum expectation values of the renormalized creation operators. In Section 4 decay properties of these vacuum expectation values are derived with the help of the method of non-stationary phase. Important input in this section are our spectral results proven in [DP12] and summarized in Subsection 1.2. The more technical part of our discussion is postponed to appendices.

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1.1 The model

We consider an interacting system of massive spinless bosons, which we will call ‘electrons’, and massless spinless bosons, which we will call ‘photons’. Let $\mathfrak{h}_e := L^2(\mathbb{R}^3, d^3p)$ and $\mathfrak{h}_f := L^2(\mathbb{R}^3, d^3k)$ be the single-electron and single-photon spaces, respectively, and let $\Gamma(\mathfrak{h}_e)$ and $\Gamma(\mathfrak{h}_f)$ be the corresponding symmetric Fock spaces. The (improper) creation and annihilation operators on $\Gamma(\mathfrak{h}_e)$ (resp. $\Gamma(\mathfrak{h}_f)$) will be denoted by $\eta^*(p), \eta(p)$ (resp. $a^*(k), a(k)$). They satisfy the canonical commutation relations:

$$[\eta(p), \eta^*(p')] = \delta(p - p'), \quad [\eta(p), \eta(p')] = [\eta^*(p), \eta^*(p')] = 0, \quad (1.17)$$

$$[a(k), a^*(k')] = \delta(k - k'), \quad [a(k), a(k')] = [a^*(k), a^*(k')] = 0. \quad (1.18)$$

The free Hamiltonians of electrons and photons are given by

$$H_e := \int d^3p \Omega(p) \eta^*(p) \eta(p), \quad H_f := \int d^3k \omega(k) a^*(k) a(k), \quad (1.19)$$

where $\Omega(p) = \frac{p^2}{2}$ and $\omega(k) = |k|$. We recall that these operators are essentially self-adjoint on C_e, C_f , respectively, where $C_{e/f} \subset \Gamma(\mathfrak{h}_{e/f})$ are dense subspaces consisting of finite linear combinations of symmetrized tensor products of elements of $C_0^\infty(\mathbb{R}^3)$.

The physical Hilbert space of our system is $\mathcal{H} := \Gamma(\mathfrak{h}_e) \otimes \Gamma(\mathfrak{h}_f)$ and we will follow the standard convention to denote operators of the form $A \otimes 1$ and $1 \otimes B$ by A and B , respectively. The Hamiltonian describing the free evolution of the composite system of electrons and photons is given by

$$H_{\text{fr}} := H_e + H_f \quad (1.20)$$

and it is essentially self-adjoint on $C := C_e \otimes C_f$. Now let us introduce the interaction between electrons and photons. Let $\lambda > 0$ be the coupling constant, $\kappa = 1$ be the ultraviolet cut-off² and let $1/2 \geq \bar{\alpha} \geq 0$ be a parameter which controls the infrared behavior of the system. Given these parameters, we define the form-factor

$$v_{\bar{\alpha}}(k) := \lambda \frac{\chi_\kappa(k) |k|^{\bar{\alpha}}}{(2|k|)^{\frac{1}{2}}}, \quad (1.21)$$

²We set $\kappa = 1$ to simplify the proofs of Proposition 1.1 and Theorem 1.2, given in the companion paper [DP12]. In the present paper we will write κ explicitly.

where $\chi_\kappa \in C_0^\infty(\mathbb{R}^3)$ is rotationally invariant, non-increasing in the radial direction, supported in \mathcal{B}_κ and equal to one on $\mathcal{B}_{(1-\varepsilon_0)\kappa}$, for some fixed $0 < \varepsilon_0 < 1$. (We denote by \mathcal{B}_r the open ball of radius r centered at zero).

The interaction Hamiltonian, defined as a symmetric operator on C , is given by the following formula

$$H_I := \int d^3 p d^3 k v_{\bar{\alpha}}(k) \eta^*(p+k) a(k) \eta(p) + \text{h.c.} \quad (1.22)$$

For future reference we denote by H_I^a be the first term on the r.h.s. of (1.22) and set $H_I^c = (H_I^a)^*$.

As indicated in [Fr73, Fr74], the full Hamiltonian $H = H_{\text{fr}} + H_I$ can be defined as a self-adjoint operator on a dense domain in \mathcal{H} . For the reader's convenience we outline briefly this construction: First, we note that both H_{fr} and H_I preserve the number of electrons. Let us therefore define $\mathcal{H}^{(n)} := \Gamma^{(n)}(\mathfrak{h}_e) \otimes \Gamma(\mathfrak{h}_f)$, where $\Gamma^{(n)}(\mathfrak{h}_e)$ is the n -particle subspace of $\Gamma(\mathfrak{h}_e)$ and let $H_{\text{fr}}^{(n)}$ and $H_I^{(n)}$ be the restrictions of the respective operators to $\mathcal{H}^{(n)}$, defined of $C^{(n)} := C \cap \mathcal{H}^{(n)}$. As shown in Lemma A.1, using the Kato-Rellich theorem, each $H^{(n)} = H_{\text{fr}}^{(n)} + H_I^{(n)}$ can be defined as a bounded from below, self-adjoint operator on the domain of $H_{\text{fr}}^{(n)}$, which is essentially self-adjoint on $C^{(n)}$. Then we can define

$$H := \bigoplus_{n \in \mathbb{N}_0} H^{(n)} \quad (1.23)$$

as an operator on C . Since $H^{(n)} \pm i$ have dense ranges on $C^{(n)}$, $H \pm i$ has a dense range on C , thus it is essentially self-adjoint on this domain. We stress that the above construction is valid both in the infrared-regular case ($1/2 \geq \bar{\alpha} > 0$) and in the infrared singular situation ($\bar{\alpha} = 0$).

On C we have the following formula for H

$$\begin{aligned} H = \int d^3 p \Omega(p) \eta^*(p) \eta(p) &+ \int d^3 k \omega(k) a^*(k) a(k) \\ &+ \left(\int d^3 p d^3 k v_{\bar{\alpha}}(k) \eta^*(p+k) a(k) \eta(p) + \text{h.c.} \right). \end{aligned} \quad (1.24)$$

It reduces to a more familiar expression on $C^{(n)}$

$$H^{(n)} = \sum_{i=1}^n \frac{(i \nabla_{x_i})^2}{2} + \int d^3 k \omega(k) a^*(k) a(k) + \sum_{i=1}^n \int d^3 k v_{\bar{\alpha}}(k) (e^{ikx_i} a(k) + e^{-ikx_i} a^*(k)), \quad (1.25)$$

where x_i are the position operators of the electrons. Finally, we introduce the electron and photon momentum operators

$$P_e^i := \int d^3 p p^i \eta^*(p) \eta(p), \quad P_f^i := \int d^3 k k^i a^*(k) a(k), \quad i \in \{1, 2, 3\}, \quad (1.26)$$

which are essentially self-adjoint on C . We recall that H is translationally invariant, that is it commutes with the total momentum operators P^i , given by

$$P^i := P_e^i + P_f^i, \quad i \in \{1, 2, 3\}, \quad (1.27)$$

which are also essentially self-adjoint on C .

1.2 Spectral theory

In this section we collect some necessary information about the spectrum of the single-electron Hamiltonian $H^{(1)}$. We recall that the analysis of the spectrum of $H_p^{(1)}$ was initiated in [Fr73, Fr74] and advanced in [Pi03] with the help of the iterative analytic perturbation theory. Further developments along these lines, which we describe in detail below, can be found in [DP12]. Interesting results on the spectrum of the Nelson model with a slightly different form factor were also obtained in [AH12] by different methods.

We recall that due to the translational invariance $H^{(1)}$ can be decomposed into a direct integral of the fiber Hamiltonians $H_p^{(1)}$ as follows

$$H^{(1)} = \Pi^* \int^{\oplus} d^3 P H_p^{(1)} \Pi. \quad (1.28)$$

Here $\Pi : \mathcal{H}^{(1)} \rightarrow L^2(\mathbb{R}^3; \Gamma(\mathfrak{h}_{\text{fi}}))$ is a unitary map given by $\Pi := F e^{iP_{\text{f}} x}$, where x^i are the electron's position operators and F is the Fourier transform in the electron's variables. The Hamiltonians $H_p^{(1)}$ are self-adjoint operators on the Fock space $\Gamma(\mathfrak{h}_{\text{fi}})$, where $\mathfrak{h}_{\text{fi}} = L^2(\mathbb{R}^3, d^3 k)$ is the single-photon space in the fiber picture. Denoting by $b^*(k)$ and $b(k)$ the creation and annihilation operators on $\Gamma(\mathfrak{h}_{\text{fi}})$ one easily obtains from (1.25) that

$$H_p^{(1)} = \frac{1}{2}(P - P_{\text{f}})^2 + H_{\text{f}} + \int d^3 k v_{\vec{\alpha}}(k) (b(k) + b^*(k)). \quad (1.29)$$

As a tool to study the spectrum of this Hamiltonian, we introduce auxiliary fiber Hamiltonians with infrared cut-offs

$$H_{p,\sigma}^{(1)} = \frac{1}{2}(P - P_{\text{f}})^2 + H_{\text{f}} + \int d^3 k v_{\vec{\alpha}}^{\sigma}(k) (b(k) + b^*(k)). \quad (1.30)$$

The form-factor $v_{\vec{\alpha}}^{\sigma}$ is defined as follows

$$v_{\vec{\alpha}}^{\sigma}(k) := \lambda \frac{\chi_{[\sigma,\kappa)}(k) |k|^{\vec{\alpha}}}{(2|k|)^{\frac{1}{2}}}, \quad (1.31)$$

where $0 < \sigma \leq \kappa$, $\chi_{[\sigma,\kappa)}(k) := \mathbf{1}_{\mathcal{B}'_{\sigma}}(k) \chi_{\kappa}(k)$, \mathcal{B}'_{σ} is the complement of the ball of radius σ and $\mathbf{1}_{\Delta}$ is the characteristic function of a set Δ . By replacing $v_{\vec{\alpha}}$ with $v_{\vec{\alpha}}^{\sigma}$ in (1.22), we obtain the interaction Hamiltonian $H_{\text{I},\sigma}$ and the corresponding full Hamiltonian H_{σ} with an infrared cut-off. The restriction of H_{σ} to the single-electron subspace, denoted $H_{\sigma}^{(1)}$, has a fiber decomposition into the Hamiltonians $H_{p,\sigma}^{(1)}$.

Since we are particularly interested in the bottom of the spectrum of $H_p^{(1)}$ and $H_{p,\sigma}^{(1)}$, let us define

$$E_p := \inf \sigma(H_p^{(1)}), \quad E_{p,\sigma} := \inf \sigma(H_{p,\sigma}^{(1)}). \quad (1.32)$$

As the model is non-relativistic, we restrict attention to small values of the total momentum P at which the electron moves slower than the photons. More precisely, we consider P from the set

$$S := \{ P \in \mathbb{R}^3 \mid |P| < P_{\text{max}} \} \quad (1.33)$$

for some $P_{\text{max}} > 0$. Since we work in the weak coupling regime, we fix some sufficiently small $\lambda_0 > 0$, and restrict attention to $\lambda \in (0, \lambda_0]$. The parameters P_{max} and λ_0 are specified

in Proposition 1.1 and Theorem 1.2 below. P_{\max} remains fixed in the course of our analysis. The maximal coupling constant λ_0 is readjusted only in the last step of our investigation – in Theorem 2.1 – to a new value which is denoted λ'_0 .

In the following proposition we collect the results concerning $E_P, E_{P,\sigma}$ which will be needed in the present investigation. Many of these properties are known, but several are new, as we explain below.

Proposition 1.1. *Fix $0 \leq \bar{\alpha} \leq 1/2$ and let $P_{\max} = 1/6$. Then there exists $\lambda_0 > 0$ s.t. for all $P \in S := \mathcal{B}_{P_{\max}}$, $\lambda \in (0, \lambda_0]$ there holds:*

- (a) *$S \ni P \mapsto E_P$ is twice continuously differentiable and strictly convex. $S \ni P \mapsto E_{P,\sigma}$ is analytic and strictly convex, uniformly in $\sigma \in (0, \kappa]$. Moreover,*

$$|E_P - E_{P,\sigma}| \leq c\sigma, \quad (1.34)$$

$$|\partial_P^{\beta_1} E_{P,\sigma}| \leq c, \quad |\partial_P^{\beta_2} E_{P,\sigma}| \leq c, \quad |\partial_P^{\beta_3} E_{P,\sigma}| \leq c/\sigma^{\delta_{\lambda_0}} \quad (1.35)$$

for multiindices β_j s.t. $|\beta_j| = j$, $j \in \{1, 2, 3\}$.

- (b) *For $\sigma > 0$, $E_{P,\sigma}$ is a simple eigenvalue corresponding to a normalized eigenvector $\psi_{P,\sigma}$, whose phase is specified in [DP12]. There holds*

$$\|\partial_P^\beta \psi_{P,\sigma}\| \leq c/\sigma^{\delta_{\lambda_0}} \quad (1.36)$$

for multiindices β s.t. $0 < |\beta| \leq 2$.

- (c) *For $\bar{\alpha} > 0$, E_P is a simple eigenvalue corresponding to a normalized eigenvector ψ_P . Moreover, for a suitable choice of the phase of ψ_P ,*

$$\|\psi_P - \psi_{P,\sigma}\| \leq c\sigma^{\bar{\alpha}}. \quad (1.37)$$

The constant c above is independent of σ , P , λ , $\bar{\alpha}$ within the assumed restrictions. Clearly, all statements above remain true after replacing λ_0 by some $\tilde{\lambda}_0 \in (0, \lambda_0]$. The resulting function $\tilde{\lambda}_0 \mapsto \delta_{\tilde{\lambda}_0}$ can be chosen positive and s.t. $\lim_{\tilde{\lambda}_0 \rightarrow 0} \delta_{\tilde{\lambda}_0} = 0$.

In the light of Proposition 1.1 we can define the subspace of renormalized single-electron states

$$\mathcal{H}_{1,\sigma} := \{\Pi^* \int^\oplus d^3 P h(P) \psi_{P,\sigma} \mid h \in L^2(\mathbb{R}^3, d^3 P), \text{supp } h \subset S\}. \quad (1.38)$$

In the case of $1/2 \geq \bar{\alpha} > 0$ we also set $\mathcal{H}_1 := \mathcal{H}_{1,\sigma=0}$.

Large part of Proposition 1.1 has already been established in the Nelson model or in similar models: The fact that $S \ni P \mapsto E_P$ is twice continuously differentiable and convex has been shown in non-relativistic and semi-relativistic QED in [FP10, KM12, BCFS07] and in the Nelson model with a slightly different form-factor in [AH12]. The present case is covered by our analysis in [DP12]. The bound (1.34) can be extracted from [Pi03]. First statement in (b) has been established already in [Fr74]. Part (c) is implicit in [Pi03] and is shown explicitly in [DP12]. The bound on the third derivative of $E_{P,\sigma}$ in (1.35) and on the first and second derivative of $\psi_{P,\sigma}$ in (1.36) are new and are proven in [DP12].

It turns out that the properties stated in Proposition 1.1 are not quite enough for our purposes. Scattering theory for several electrons requires much more detailed information about

the electron's localization in space than scattering of one electron and photons. This information is contained in regularity properties of the momentum wave functions of the vectors $\psi_{P,\sigma}$. Let us express $\psi_{P,\sigma}$ in terms of its m -particle components in the Fock space:

$$\psi_{P,\sigma} = \{f_{P,\sigma}^m\}_{m \in \mathbb{N}_0}, \quad (1.39)$$

where $f_{P,\sigma}^m \in L_{\text{sym}}^2(\mathbb{R}^{3m}, d^{3m}k)$, i.e., each $f_{P,\sigma}^m$ is a square-integrable function symmetric in m variables from \mathbb{R}^3 . Let us introduce the following auxiliary functions:

$$g_\sigma^m(k_1, \dots, k_m) := \prod_{i=1}^m \frac{c \lambda \chi_{[\sigma, \kappa_*]}(k_i) |k_i|^{\bar{\alpha}}}{|k_i|^{3/2}}, \quad \kappa_* := (1 - \varepsilon_0)^{-1} \kappa, \quad 0 < \varepsilon_0 < 1, \quad (1.40)$$

where c is some positive constant independent of m, σ, P and λ within the restrictions specified above. Finally, we introduce the notation

$$\mathcal{A}_{r_1, r_2} := \{k \in \mathbb{R}^3 \mid r_1 < |k| < r_2\}, \quad (1.41)$$

where $0 \leq r_1 < r_2$. Now we are ready to state the required properties of the functions $f_{P,\sigma}^m$:

Theorem 1.2. *Fix $0 \leq \bar{\alpha} \leq 1/2$ and set $P_{\max} = 1/6$. Then there exists $\lambda_0 > 0$ s.t. for all $P \in S = \mathcal{B}_{P_{\max}}$, $\lambda \in (0, \lambda_0]$ there holds:*

- (a) *Let $\{f_{P,\sigma}^m\}_{m \in \mathbb{N}_0}$ be the m -particle components of $\psi_{P,\sigma}$ and let $\overline{\mathcal{A}}_{\sigma, \kappa}^{\times m}$ be defined by (1.41). Then, for any $P \in S$, the function $f_{P,\sigma}^m$ is supported in $\overline{\mathcal{A}}_{\sigma, \kappa}^{\times m}$.*

- (b) *The function*

$$S \times \mathcal{A}_{\sigma, \infty}^{\times m} \ni (P; k_1, \dots, k_m) \mapsto f_{P,\sigma}^m(k_1, \dots, k_m) \quad (1.42)$$

is twice continuously differentiable and extends by continuity, together with its derivatives, to the set $S \times \overline{\mathcal{A}}_{\sigma, \infty}^{\times m}$.

- (c) *For any multiindex β , $0 \leq |\beta| \leq 2$, the function (1.42) satisfies*

$$|\partial_{k_l}^\beta f_{P,\sigma}^m(k_1, \dots, k_m)| \leq \frac{1}{\sqrt{m}!} |k_l|^{-|\beta|} g_\sigma^m(k_1, \dots, k_m), \quad (1.43)$$

$$|\partial_P^\beta f_{P,\sigma}^m(k_1, \dots, k_m)| \leq \frac{1}{\sqrt{m}!} \left(\frac{1}{\sigma^{\delta_{\delta_0}}}\right)^{|\beta|} g_\sigma^m(k_1, \dots, k_m), \quad (1.44)$$

$$|\partial_{P^{i'}} \partial_{k_l} f_{P,\sigma}^m(k_1, \dots, k_m)| \leq \frac{1}{\sqrt{m}!} \frac{1}{\sigma^{\delta_{\delta_0}}} |k_l|^{-1} g_\sigma^m(k_1, \dots, k_m), \quad (1.45)$$

where the function $\tilde{\lambda}_0 \mapsto \lambda_{\tilde{\delta}_0}$ has the properties specified in Proposition 1.1.

Parts (a), (b) of Theorem 1.2 and estimate (1.43) in (c) can be extracted from [Fr73, Fr74, Fr] or proven using the methods from these papers. The key new input are the bounds (1.44), (1.45) on the first and second derivative w.r.t. P with their mild dependence on the infrared cut-off σ . These bounds require major refinements of the iterative multiscale analysis from [Pi03] and they constitute the main technical result of the companion paper [DP12].

1.3 Scattering theory

In this section we outline the construction of two-electron scattering states along the lines of Haag-Ruelle scattering theory [Ha58, Ru62], following some ideas from [Fr, Al73]. Since we are interested here in the infrared-regular situation, we set $1/2 \geq \bar{\alpha} > 0$. Such $\bar{\alpha}$ will be kept fixed in the remaining part of the paper. (The infrared-singular case $\bar{\alpha} = 0$ is much more involved due to the infraparticle problem and will be studied elsewhere).

Let $C_0^2(S)$ be the class of twice continuously differentiable functions with compact support contained in S . Following [Fr, Al73], for any $h \in C_0^2(S)$ we define the renormalized creation operator

$$\hat{\eta}_\sigma^*(h) := \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \int d^3p d^3m k h(p) f_{p,\sigma}^m(k) a^*(k)^m \eta^*(p - \underline{k}). \quad (1.46)$$

In this expression we use the following short-hand notation, which will appear frequently below:

$$f_{p,\sigma}^m(k) := f_{p,\sigma}^m(k_1, \dots, k_m), \quad (1.47)$$

$$a^*(k)^m := a^*(k_1) \dots a^*(k_m), \quad (1.48)$$

$$\underline{k} := k_1 + \dots + k_m. \quad (1.49)$$

It is shown in Lemma A.2 that $\hat{\eta}_\sigma^*(h)$ and $\hat{\eta}_\sigma^*(h_1)\hat{\eta}_\sigma^*(h_2)$, for $h_1, h_2 \in C_0^2(S)$, are well defined operators on C . (Since $\hat{\eta}_\sigma(h) := (\hat{\eta}_\sigma^*(h))^*$ is obviously well defined on C , we obtain that $\hat{\eta}_\sigma^*(h)$ is closable).

Now let Ω_e and Ω_f be the vacuum vectors of $\Gamma(\mathfrak{h}_e)$ and $\Gamma(\mathfrak{h}_f)$, respectively. Then $\Omega := \Omega_e \otimes \Omega_f$ is the physical vacuum in \mathcal{H} . As shown in Lemma 2.3 below,

$$\psi_{h,\sigma} := \hat{\eta}_\sigma^*(h)\Omega = \Pi^* \int^{\oplus} d^3P h(P) \psi_{P,\sigma}, \quad (1.50)$$

that is $\psi_{h,\sigma}$ is an element of the subspace of renormalized single-electron states $\mathcal{H}_{1,\sigma}$, defined in (1.38). Since we assumed that $1/2 \geq \bar{\alpha} > 0$, we obtain from Proposition 1.1 (c) that there exists the limit

$$\psi_h := \lim_{\sigma \rightarrow 0} \hat{\eta}_\sigma^*(h)\Omega = \Pi^* \int^{\oplus} d^3P h(P) \psi_P. \quad (1.51)$$

Clearly, ψ_h belongs to the renormalized single-electron space \mathcal{H}_1 of the Hamiltonian H .

Let us now proceed to the construction of two-electron scattering states. We fix some parameter $\gamma_0 > 4$, which will be fixed in our investigation, choose some $\gamma \in (4, \gamma_0]$ and introduce a time-dependent cut-off

$$\sigma_t := \kappa/t^\gamma \quad (1.52)$$

for $t \geq \max\{1, \kappa\}$. Next, we choose $h_1, h_2 \in C_0^2(S)$, with disjoint supports and set

$$h_{i,t}(p) := e^{-iE_p t} h_i(p), \quad i \in \{1, 2\}. \quad (1.53)$$

Now we are ready to define the two-electron scattering states approximants:

$$\Psi_{t,h_1,h_2} := e^{iHt} \hat{\eta}_{\sigma_t}^*(h_{1,t}) \hat{\eta}_{\sigma_t}^*(h_{2,t}) \Omega. \quad (1.54)$$

We will show that the limit of Ψ_{t,h_1,h_2} exists as $t \rightarrow \infty$ and can be interpreted as a physical state describing two independent excitations. This is the content of our main result concerning scattering theory, stated below.

Theorem 1.3. Fix $1/2 \geq \bar{\alpha} > 0$, $\gamma_0 > 4$. Let $\lambda \in (0, \lambda'_0]$, where $\lambda'_0 > 0$ is sufficiently small. Then, for $h_1, h_2 \in C_0^2(S)$ with disjoint supports, the following statements hold:

(a) Let $\sigma_t = \kappa/t^\gamma$, where $\gamma \in (4, \gamma_0]$. Then there exists the limit

$$\Psi_{h_1, h_2}^+ := \lim_{t \rightarrow \infty} e^{iHt} \hat{\eta}_{\sigma_t}^*(h_{1,t}) \hat{\eta}_{\sigma_t}^*(h_{2,t}) \Omega \quad (1.55)$$

and it is called the two-electron scattering state. It is independent of the parameter γ within the above restrictions.

(b) Let Ψ_{h_1, h_2}^+ , $\Psi_{h'_1, h'_2}^+$ be two scattering states. Their scalar product has the form

$$\langle \Psi_{h_1, h_2}^+, \Psi_{h'_1, h'_2}^+ \rangle = \langle \psi_{h_1}, \psi_{h'_1} \rangle \langle \psi_{h_2}, \psi_{h'_2} \rangle + \langle \psi_{h_1}, \psi_{h'_2} \rangle \langle \psi_{h_2}, \psi_{h'_1} \rangle. \quad (1.56)$$

Proof. Part (a) follows from Theorem 2.1. Assumptions (2.2), (2.3) of this theorem are verified in Proposition 3.4. Assumption 2.4 follows from Propositions 2.2, 3.1, 3.6 and Corollary 3.5. Part (b) of the theorem follows from Proposition 3.4 and Proposition 1.1 (c). \square

We remark that an essential ingredient of the proof of convergence in (1.55) is the disjointness of velocity supports of the functions h_1, h_2 , defined as

$$V(h_i) := \{ \nabla E_p \mid p \in \text{supp } h_i \}, \quad i \in \{1, 2\}. \quad (1.57)$$

This property follows from the assumed disjointness of the supports of h_1, h_2 and the invertibility of the relation $S \ni p \mapsto E_p$, guaranteed by Proposition 1.1 (a).

We note that the vector ψ_h , given by (1.51), is determined uniquely by the function $h \in C_0^2(S)$. Moreover, vectors of the form $\psi_{h_1} \otimes_s \psi_{h_2}$, where $h_1, h_2 \in C_0^2(S)$ have disjoint supports, and \otimes_s is the symmetric tensor product, span a dense subspace in $\mathcal{H}_1 \otimes_s \mathcal{H}_1$. Thus we can define the wave operator $W^+ : \mathcal{H}_1 \otimes_s \mathcal{H}_1 \rightarrow \mathcal{H}$ as follows

$$W^+(\psi_{h_1} \otimes_s \psi_{h_2}) := \Psi_{h_1, h_2}^+. \quad (1.58)$$

By Theorem 1.3 (b), this map is an isometry.

Standing assumptions and conventions:

1. The parameters $P_{\max} = 1/6$ and $1/2 \geq \bar{\alpha} > 0$ are kept fixed in the remaining part of the paper.
2. The maximal coupling constant $\lambda_0 > 0$ is as specified in Proposition 1.1 and Theorem 1.2 for the values of P_{\max} and $1/2 \geq \bar{\alpha} > 0$ fixed above. It remains unchanged in Sections 3 and 4. It is readjusted to a possibly smaller value $\lambda'_0 > 0$ in the last step of the analysis in Theorem 2.1. λ'_0 may depend of γ_0 but not on $\gamma \in (4, \gamma_0]$.
3. $\tilde{\lambda}_0 \mapsto \delta_{\tilde{\lambda}_0}$, $\tilde{\lambda}_0 \mapsto \delta'_{\tilde{\lambda}_0}$, will denote positive functions of $\tilde{\lambda}_0 \in (0, \lambda_0]$, which may differ from line to line, and have the property

$$\lim_{\tilde{\lambda}_0 \rightarrow 0} \delta_{\tilde{\lambda}_0} = 0, \quad \lim_{\tilde{\lambda}_0 \rightarrow 0} \delta'_{\tilde{\lambda}_0} = 0. \quad (1.59)$$

Such functions control the infrared behaviour of our estimates listed in the statement of Theorem 2.1. By reducing the coupling constant from λ_0 to λ'_0 we make this behaviour sufficiently mild.

4. We denote by $\gamma \in (4, \gamma_0]$ the parameter which controls the time dependence of the (fast) infrared cut-off i.e., $\sigma_t = \kappa/t^\gamma$. The parameter γ_0 is kept fixed in the remaining part of the paper. This parameter appears also in the definition of the slow infrared cut-off $\sigma_s = \kappa(\sigma/\kappa)^{1/(8\gamma_0)}$ in the proofs of Lemmas 4.1 and 4.2.
5. We will denote by c, c', c'' numerical constants which may depend on $S, \lambda_0, \varepsilon_0, \kappa, \bar{\alpha}, \gamma_0$ and functions h_1, h_2 but not on σ, t or the electron and photon momenta. The values of these constants may change from line to line.
6. We will denote by $(p, q) \mapsto D(p, q), (p, q) \mapsto D'(p, q)$ smooth, compactly supported functions on $\mathbb{R}^3 \times \mathbb{R}^3$, which may depend of $S, \lambda_0, \varepsilon_0, \kappa, \bar{\alpha}$ but not on σ, t or photon momenta.
7. We will denote by $k = (k_1, \dots, k_m) \in \mathbb{R}^{3m}$ a collection of photon variables. A lower or upper index m of a function indicates that it is a symmetric function of (k_1, \dots, k_m) . For example:

$$f^m(k) := f^m(k_1, \dots, k_m). \quad (1.60)$$

Similarly, we set $a^*(k)^m := a^*(k_1) \dots a^*(k_m)$. We note that the order in which the components of k are listed is irrelevant, since they enter always into symmetric expressions.

8. We separate the electron variable $p \in \mathbb{R}^3$ and the photon variables $k \in \mathbb{R}^{3m}$ by a semi-colon. For example:

$$G_m(p; k) := G_m(p; k_1, \dots, k_m). \quad (1.61)$$

9. Two collections of electron and photon variables $p \in \mathbb{R}^3, k \in \mathbb{R}^{3m}$ and $q \in \mathbb{R}^3, r \in \mathbb{R}^{3n}$ are separated by a bar. For example

$$F_{m,n}(p; k | q; r) := F_{m,n}(p; k_1, \dots, k_m | q; r_1, \dots, r_n). \quad (1.62)$$

10. Given $k = (k_1, \dots, k_m)$ we write $\underline{k} := k_1 + \dots + k_m$.

2 Main ingredients of the proof

Theorem 2.1 below gives the existence of scattering states based on some assumptions proven in the later part of this paper: Property (2.2) follows from Proposition 3.4. To establish property (2.4) we derive a formula for time-derivatives of scattering state approximants in Proposition 2.2 in the later part of this section. The section concludes with Lemma 2.3 which gives a direct integral representation of the vectors $\psi_{h,\sigma} = \hat{\eta}_\sigma^*(h)\Omega$. We used this result already in (1.46) above.

In our proof of convergence of the scattering state approximants (1.54) we will vary time t and the infrared cut-off σ independently. Let us therefore fix $h_1, h_2 \in C_0^2(S)$ with disjoint supports and introduce an auxiliary two-parameter sequence

$$\Psi_{t,\sigma} := e^{iHt} \hat{\eta}_\sigma^*(h_{1,t}) \hat{\eta}_\sigma^*(h_{2,t}) \Omega. \quad (2.1)$$

Now we are ready to state and prove the main result of this section.

Theorem 2.1. Let $h_1, h_2 \in C_0^2(S)$ have disjoint supports, let $\Psi_{t,\sigma}$ be given by (2.1). Suppose that for $\lambda \in (0, \lambda_0]$, infrared cut-offs σ, σ' s.t. $\sigma \leq \sigma' \leq \kappa$ and $t \geq \max\{1, \kappa\}$

$$\langle \Psi_{t,\sigma'}, \Psi_{t,\sigma} \rangle = \langle \psi_{h_1,\sigma'}, \psi_{h_1,\sigma} \rangle \langle \psi_{h_2,\sigma'}, \psi_{h_2,\sigma} \rangle + R(t, \sigma, \sigma'), \quad (2.2)$$

$$|R(t, \sigma, \sigma')| \leq c \frac{1}{\sigma^{\delta_{\lambda_0}}} \left(\frac{1}{t} \frac{1}{\sigma^{1/(8\gamma_0)}} + (\sigma')^{2\bar{\alpha}} \right), \quad (2.3)$$

$$\|\partial_t \Psi_{t,\sigma}\| \leq \frac{c}{\sigma^{\delta_{\lambda_0}}} \left(\frac{\sigma^{\bar{\alpha}/(4\gamma_0)}}{t} + \sigma t + \frac{1}{t^2 \sigma^{1/(4\gamma_0)}} \right) + c \sigma^{1-\delta_{\lambda_0}} + c \sigma^{1/2-\delta_{\lambda_0}} \left(1 + \frac{1}{t} \frac{1}{\sigma^{1/(8\gamma_0)}} \right). \quad (2.4)$$

Then one can choose $\lambda'_0 \in (0, \lambda_0]$ s.t. for $\lambda \in (0, \lambda'_0]$ there exists the limit

$$\Psi_{h_1, h_2}^+ = \lim_{t \rightarrow \infty} \Psi_{t, \sigma_t}, \quad (2.5)$$

where $\sigma_t = \kappa/t^\gamma$. This limit is independent of the choice of $\gamma \in (4, \gamma_0]$.

Proof. We assume that $t_2 \geq t_1 \geq 1$ are sufficiently large so that $\sigma_{t_2} \leq \sigma_{t_1} \leq 1$. We write

$$\|\Psi_{t_2, \sigma_{t_2}} - \Psi_{t_1, \sigma_{t_1}}\| \leq \|\Psi_{t_2, \sigma_{t_2}} - \Psi_{t_1, \sigma_{t_2}}\| + \|\Psi_{t_1, \sigma_{t_2}} - \Psi_{t_1, \sigma_{t_1}}\|. \quad (2.6)$$

Concerning the first term on the r.h.s. of (2.6), we note that the bound in (2.4) implies

$$\|\partial_t \Psi_{t, \sigma_{t_2}}\| \leq \frac{c}{\sigma_{t_2}^{\delta_{\lambda_0}}} \left(\frac{\sigma_{t_2}^\varepsilon}{t} + \frac{1}{t^2 \sigma_{t_2}^{1/(4\gamma_0)}} \right) + c \sigma_{t_2}^{1/2-\delta_{\lambda_0}} t \quad (2.7)$$

for some $\varepsilon > 0$, depending on γ_0 , but independent of λ_0 . Now we estimate

$$\begin{aligned} \|\Psi_{t_2, \sigma_{t_2}} - \Psi_{t_1, \sigma_{t_2}}\| &\leq \int_{t_1}^{t_2} dt \|\partial_t \Psi_{t, \sigma_{t_2}}\| \leq c \sigma_{t_2}^{\varepsilon-\delta_{\lambda_0}} \log(t_2/t_1) + c \frac{1}{t_1} \frac{1}{\sigma_{t_2}^{\delta_{\lambda_0}+1/(4\gamma_0)}} + c \sigma_{t_2}^{1/2-\delta_{\lambda_0}} t_2^2 \\ &\leq c \sigma_{t_2}^{\varepsilon'} + c \frac{1}{t_1} \frac{1}{\sigma_{t_2}^{1/(3\gamma_0)}} + c \sigma_{t_2}^{1/2-\delta_{\lambda_0}} t_2^2 \leq c \frac{1}{t_2^{\varepsilon''}} + c \frac{t_2^{1/3}}{t_1} \\ &\leq c \left(\frac{t_2^{1/3}}{t_1} + \left(\frac{t_2^{1/3}}{t_1} \right)^{\varepsilon''} \right), \end{aligned} \quad (2.8)$$

where in the third step we made use of the fact that $t_2^{-\alpha} \log(t_2/t_1)$ is uniformly bounded in $t_2 \geq t_1 \geq 1$ for any $\alpha > 0$ and chose λ_0 sufficiently small (depending on γ_0) to ensure that $\varepsilon' > 0$ and that $\delta_{\lambda_0} + 1/(4\gamma_0) \leq 1/(3\gamma_0)$. In the fourth step we chose λ_0 again sufficiently small and exploited the fact that $\gamma > 4$ to ensure that $1 > \varepsilon'' > 0$.

As for the second term on the r.h.s. of (2.6), we get

$$\begin{aligned} &\|\Psi_{t_1, \sigma_{t_2}} - \Psi_{t_1, \sigma_{t_1}}\|^2 \\ &= \langle \Psi_{t_1, \sigma_{t_2}}, \Psi_{t_1, \sigma_{t_2}} \rangle + \langle \Psi_{t_1, \sigma_{t_1}}, \Psi_{t_1, \sigma_{t_1}} \rangle - 2\text{Re} \langle \Psi_{t_1, \sigma_{t_2}}, \Psi_{t_1, \sigma_{t_1}} \rangle \\ &= \langle \psi_{h_1, \sigma_{t_2}}, \psi_{h_1, \sigma_{t_2}} \rangle \langle \psi_{h_2, \sigma_{t_2}}, \psi_{h_2, \sigma_{t_2}} \rangle + R(t_1, \sigma_{t_2}, \sigma_{t_2}) \\ &\quad + \langle \psi_{h_1, \sigma_{t_1}}, \psi_{h_1, \sigma_{t_1}} \rangle \langle \psi_{h_2, \sigma_{t_1}}, \psi_{h_2, \sigma_{t_1}} \rangle + R(t_1, \sigma_{t_1}, \sigma_{t_1}) \\ &\quad - \langle \psi_{h_1, \sigma_{t_2}}, \psi_{h_1, \sigma_{t_1}} \rangle \langle \psi_{h_2, \sigma_{t_2}}, \psi_{h_2, \sigma_{t_1}} \rangle + \overline{R(t_1, \sigma_{t_2}, \sigma_{t_1})} \\ &\quad - \langle \psi_{h_1, \sigma_{t_1}}, \psi_{h_1, \sigma_{t_2}} \rangle \langle \psi_{h_2, \sigma_{t_1}}, \psi_{h_2, \sigma_{t_2}} \rangle + \overline{R(t_1, \sigma_{t_2}, \sigma_{t_1})}. \end{aligned} \quad (2.9)$$

Thus making use of Proposition 1.1 (c) and (2.3), we obtain

$$\begin{aligned}
\|\Psi_{t_1, \sigma_{t_2}} - \Psi_{t_1, \sigma_{t_1}}\|^2 &\leq c\sigma_{t_1}^{\bar{\alpha}} + |R(t_1, \sigma_{t_2}, \sigma_{t_2})| + |R(t_1, \sigma_{t_1}, \sigma_{t_1})| + 2|R(t_1, \sigma_{t_2}, \sigma_{t_1})| \\
&\leq c\sigma_{t_1}^{\bar{\alpha}} + c\frac{1}{\sigma_{t_2}^{\delta_{\lambda_0}}} \left(\frac{1}{t_1} \frac{1}{\sigma_{t_2}^{1/(8\gamma_0)}} + (\sigma_{t_2})^{2\bar{\alpha}} \right) \\
&\quad + c\frac{1}{\sigma_{t_1}^{\delta_{\lambda_0}}} \left(\frac{1}{t_1} \frac{1}{\sigma_{t_1}^{1/(8\gamma_0)}} + (\sigma_{t_1})^{2\bar{\alpha}} \right) \\
&\quad + c\frac{1}{\sigma_{t_2}^{\delta_{\lambda_0}}} \left(\frac{1}{t_1} \frac{1}{\sigma_{t_2}^{1/(8\gamma_0)}} + (\sigma_{t_1})^{2\bar{\alpha}} \right). \tag{2.10}
\end{aligned}$$

Thus, by choosing λ_0 sufficiently small, we can find $0 < \delta_1, \delta_2, \delta < 1$ s.t.

$$\begin{aligned}
\|\Psi_{t_1, \sigma_{t_2}} - \Psi_{t_1, \sigma_{t_1}}\|^2 &\leq c \left(\frac{1}{t_1^{\delta_1}} + \frac{t_2^{1/3}}{t_1} + \frac{1}{t_2^{\delta_2}} + \frac{t_2^{\delta_{\lambda_0}}}{t_1^{\delta_1}} \right) \\
&\leq c \left(\frac{t_2^{\delta_{\lambda_0}}}{t_1^{\delta}} + \frac{t_2^{1/3}}{t_1} \right) \leq c \left(\left(\frac{t_2^{1/3}}{t_1} \right)^{\delta} + \frac{t_2^{1/3}}{t_1} \right), \tag{2.11}
\end{aligned}$$

where in the second step we used that $t_2 \geq t_1$. Consequently

$$\|\Psi_{t_1, \sigma_{t_2}} - \Psi_{t_1, \sigma_{t_1}}\| \leq c \left(\left(\frac{t_2^{1/3}}{t_1} \right)^{\delta/2} + \left(\frac{t_2^{1/3}}{t_1} \right)^{1-\delta/2} \right). \tag{2.12}$$

In view of (2.8), we get

$$\|\Psi_{t_2, \sigma_{t_2}} - \Psi_{t_1, \sigma_{t_1}}\| \leq c \sum_{i=1}^4 \left(\frac{t_2^{1/3}}{t_1} \right)^{\varepsilon_i} \tag{2.13}$$

for $0 < \varepsilon_i \leq 1$. Let us now set $\Psi(t) := \Psi_{t, \sigma_t}$ and proceed as in the proof of Theorem 3.1 of [Pi05]: Suppose $t_1^n \leq t_2 < t_1^{n+1}$. Then we can write

$$\begin{aligned}
\|\Psi(t_2) - \Psi(t_1)\| &\leq \left(\sum_{k=1}^{n-1} \|\Psi(t_1^{k+1}) - \Psi(t_1^k)\| \right) + \|\Psi(t_2) - \Psi(t_1^n)\| \\
&\leq c \sum_{i=1}^4 \sum_{k=1}^n \left(\frac{1}{t_1^{\varepsilon_i(2k/3-1/3)}} \right) \leq c \sum_{i=1}^4 t_1^{-\varepsilon_i/3} \frac{1}{1 - (1/t_1)^{2\varepsilon_i/3}}. \tag{2.14}
\end{aligned}$$

Since the last expression tends to zero as $t_1 \rightarrow \infty$, we obtain convergence of $t \mapsto \Psi(t)$.

Finally, let us show that the limit Ψ_{h_1, h_2}^+ is independent of the choice of the parameter $\gamma \in (4, \gamma_0]$. Let $4 < \gamma' \leq \gamma$ and let $\sigma_t = \kappa/t^{\gamma}$, $\sigma'_t = \kappa/t^{\gamma'}$ so that $\sigma_t \leq \sigma'_t$. We will show that

$$\lim_{t \rightarrow \infty} \|\Psi_{t, \sigma_t} - \Psi_{t, \sigma'_t}\| = 0. \tag{2.15}$$

Similarly as in the first part of the proof, it follows from formula (2.2) and from Proposition 1.1 (c) that

$$\lim_{t \rightarrow \infty} \langle \Psi_{t, \sigma_t}, \Psi_{t, \sigma'_t} \rangle = \lim_{t \rightarrow \infty} \langle \Psi_{t, \sigma'_t}, \Psi_{t, \sigma_t} \rangle = \lim_{t \rightarrow \infty} \langle \Psi_{t, \sigma'_t}, \Psi_{t, \sigma'_t} \rangle = \langle \psi_{h_1}, \psi_{h_1} \rangle \langle \psi_{h_2}, \psi_{h_2} \rangle. \tag{2.16}$$

This concludes the proof of (2.15). \square

In Proposition 2.2 below we derive a formula for $\partial_t \Psi_{t,\sigma}$, appearing in assumption (2.4). For the purpose of this derivation we introduce a suitable domain: First, we fix $l \in \mathbb{N}_0$, $r_1, r_2 > 0$ and consider vectors of the form

$$\Psi_l^{r_1, r_2} = \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \int d^{3l} p d^{3m} k F_{l,m}(p; k) \eta^*(p)^l a^*(k)^m \Omega, \quad (2.17)$$

where $F_{l,m} \in L_{\text{sym}}^2(\mathcal{B}_{r_1}^{\times l} \times \mathcal{B}_{r_2}^{\times m}, d^{3l} p d^{3m} k)$ i.e., $F_{l,m}$ are square-integrable functions, symmetric (independently) in their electron and photon variables and supported in each electron (resp. photon) variable in a ball of radius r_1 (resp. r_2). Moreover, the norms of $F_{l,m}$ satisfy the bound

$$\|F_{l,m}\|_2 \leq \frac{c^m}{\sqrt{m!}} \quad (2.18)$$

for some $c \geq 0$, independent of m , which guarantees that the vector (2.17) is well defined. Now we set

$$\mathcal{D} := \text{Span}\{\Psi_l^{r_1, r_2} \mid l \in \mathbb{N}_0, r_1, r_2 > 0\}, \quad (2.19)$$

where Span means finite linear combinations. This domain is dense and it contains \mathcal{C} . We show in Lemma A.2 that $H_\sigma, H_e, H_f, H_1^{a/c}$ and $\hat{\eta}^*(h)$, $h \in C_0^2(S)$, are well-defined on \mathcal{D} and leave this domain invariant.

Proposition 2.2. *Let $h_1, h_2 \in C_0^2(S)$ have disjoint supports and let $\Psi_{t,\sigma}$ be given by (2.1). Then there holds*

$$\begin{aligned} \partial_t \Psi_{t,\sigma} = e^{itH} & \left\{ \frac{1}{2} i [[H_1^a, \hat{\eta}_\sigma^*(h_{1,t})], \hat{\eta}_\sigma^*(h_{2,t})] \Omega + i \hat{\eta}_\sigma^*(h_{1,t}) \check{H}_{1,\sigma}^c \hat{\eta}_\sigma^*(h_{2,t}) \Omega \right. \\ & \left. + i \hat{\eta}_\sigma^*(h_{1,t}) \hat{\eta}_\sigma^*(h_{2,t}^\sigma) \Omega \right\} + \{1 \leftrightarrow 2\}. \end{aligned} \quad (2.20)$$

The operators H_1^a and \check{H}_1^c are defined on \mathcal{C} by

$$H_1^a := \int d^3 p d^3 k v_{\vec{\alpha}}(k) \eta^*(p+k) a(k) \eta(p), \quad (2.21)$$

$$\check{H}_{1,\sigma}^c := \int d^3 p d^3 k \check{v}_{\vec{\alpha}}^\sigma(k) \eta^*(p-k) a^*(k) \eta(p), \quad (2.22)$$

where $\check{v}_{\vec{\alpha}}^\sigma(k) := \lambda \frac{\mathbf{1}_{\mathcal{B}_\sigma(k)} |k|^{\vec{\alpha}}}{(2|k|)^{1/2}}$ and $h_i^\sigma(p) = (E_{p,\sigma} - E_p) h_i(p)$.

Proof. We compute

$$\begin{aligned} \partial_t \Psi_{t,\sigma} &= e^{iHt} i H \hat{\eta}_\sigma^*(h_{1,t}) \hat{\eta}_\sigma^*(h_{2,t}) \Omega + e^{iHt} \hat{\eta}_\sigma^*(\partial_t h_{1,t}) \hat{\eta}_\sigma^*(h_{2,t}) \Omega + e^{iHt} \hat{\eta}_\sigma^*(h_{1,t}) \hat{\eta}_\sigma^*(\partial_t h_{2,t}) \Omega \\ &= e^{iHt} i H \hat{\eta}_\sigma^*(h_{1,t}) \hat{\eta}_\sigma^*(h_{2,t}) \Omega - e^{iHt} \hat{\eta}_\sigma^*(h_{2,t}) \hat{\eta}_\sigma^*(i(Eh_1)_t) \Omega - e^{iHt} \hat{\eta}_\sigma^*(h_{1,t}) \hat{\eta}_\sigma^*(i(Eh_2)_t) \Omega, \end{aligned} \quad (2.23)$$

where $(Eh_i)(p) := E_p h_i(p)$, $i = 1, 2$. The first term on the r.h.s. above is well defined by Lemma A.2. The equality

$$(\partial_t \hat{\eta}_\sigma^*(h_{1,t})) \hat{\eta}_\sigma^*(h_{2,t}) \Omega = \hat{\eta}_\sigma^*(\partial_t h_{1,t}) \hat{\eta}_\sigma^*(h_{2,t}) \Omega \quad (2.24)$$

can easily be justified with the help of Lemmas B.4 and 3.3. Now we note the following identity, which is meaningful due to Lemma A.2:

$$\begin{aligned} iH\hat{\eta}_\sigma^*(h_{1,t})\hat{\eta}_\sigma^*(h_{2,t})\Omega &= i[[H, \hat{\eta}_\sigma^*(h_{1,t})], \hat{\eta}_\sigma^*(h_{2,t})]\Omega + \hat{\eta}_\sigma^*(h_{1,t})iH\hat{\eta}_\sigma^*(h_{2,t})\Omega \\ &\quad + \hat{\eta}_\sigma^*(h_{2,t})iH\hat{\eta}_\sigma^*(h_{1,t})\Omega. \end{aligned} \quad (2.25)$$

As for the first term on the r.h.s. of (2.25), we note that $[H_{\text{fr}}, \hat{\eta}_\sigma^*(h_{1,t})]$ and $[H_1^c, \hat{\eta}_\sigma^*(h_{1,t})]$ are sums of products of creation operators and therefore commute with $\hat{\eta}_\sigma^*(h_{2,t})$. Thus we get

$$i[[H, \hat{\eta}_\sigma^*(h_{1,t})], \hat{\eta}_\sigma^*(h_{2,t})]\Omega = i[[H_1^a, \hat{\eta}_\sigma^*(h_{1,t})], \hat{\eta}_\sigma^*(h_{2,t})]\Omega. \quad (2.26)$$

As for the second term on the r.h.s. of (2.25), we obtain

$$\begin{aligned} \hat{\eta}_\sigma^*(h_{1,t})iH\hat{\eta}_\sigma^*(h_{2,t})\Omega &= \hat{\eta}_\sigma^*(h_{1,t})i(H - H_\sigma)\hat{\eta}_\sigma^*(h_{2,t})\Omega + \hat{\eta}_\sigma^*(h_{1,t})\hat{\eta}_\sigma^*(i(E_\sigma h_2)_t)\Omega \\ &= \hat{\eta}_\sigma^*(h_{1,t})i\check{H}_{1,\sigma}^c\hat{\eta}_\sigma^*(h_{2,t})\Omega + \hat{\eta}_\sigma^*(h_{1,t})\hat{\eta}_\sigma^*(i((E_\sigma - E)h_2)_t)\Omega \\ &\quad + \hat{\eta}_\sigma^*(ih_{1,t})\hat{\eta}_\sigma^*((Eh_2)_t)\Omega. \end{aligned} \quad (2.27)$$

Here in the first step we applied Lemma 2.3 and in the last step we made use of the fact that the operator

$$\check{H}_{1,\sigma}^a := \int d^3p d^3k \check{v}_\sigma^\sigma(k) \eta^*(p+k) a(k) \eta(p) \quad (2.28)$$

annihilates $\hat{\eta}_\sigma^*(h_{2,t})\Omega$ due to the fact that \check{v}_σ^σ is supported below the infrared cut-off. As the last term on the r.h.s. of (2.25) can be treated analogously, this concludes the proof. \square

Lemma 2.3. *Let $h \in C_0^2(S)$ and let us consider the vector $\psi_{h,\sigma} := \hat{\eta}_\sigma^*(h)\Omega$. Then*

$$\psi_{h,\sigma} = \Pi^* \int^\oplus dP h(P) \psi_{P,\sigma}. \quad (2.29)$$

Consequently, $H_\sigma \psi_{h,\sigma} = \psi_{E_\sigma h, \sigma}$, where $(E_\sigma h)(p) := E_{p,\sigma} h(p)$.

Proof. The m -particle components of $\psi_{h,\sigma}$, with the electron's variables in the configuration space representation, have the form

$$\begin{aligned} \psi_{h,\sigma}^m(x, k_1, \dots, k_m) &= \frac{1}{(2\pi)^{3/2}} \int d^3p e^{ipx} h(p + \underline{k}) f_{p+\underline{k}, \sigma}^m(k_1, \dots, k_m) \\ &= \frac{1}{(2\pi)^{3/2}} e^{-ikx} \int d^3p e^{ipx} h(p) f_{p,\sigma}^m(k_1, \dots, k_m). \end{aligned} \quad (2.30)$$

Consequently, the m -particle components of $\Pi(\psi_{h,\sigma})$ are

$$\begin{aligned} \Pi(\psi_{h,\sigma})^m(P, k_1, \dots, k_m) &= \frac{1}{(2\pi)^3} \int d^3x e^{-iPx} \int d^3p e^{ipx} h(p) f_{p,\sigma}^m(k_1, \dots, k_m) \\ &= h(P) f_{P,\sigma}^m(k_1, \dots, k_m), \end{aligned} \quad (2.31)$$

which concludes the proof. \square

3 Vacuum expectation values of renormalized creation operators

In this section we estimate the norms of the terms appearing on the r.h.s. of (2.20) in order to verify assumption (2.4) in Theorem 2.1. We also derive estimates on the norms of scattering states, required in this theorem. A crucial input is provided by the non-stationary phase analysis in Section 4, which in turn relies on the spectral information from Theorem 1.2, proven in [DP12].

In this section we will use the following definitions:

$$G_{i,m}(q; k) := e^{-iE_q t} h_i(q) f_{q,\sigma}^m(k), \quad i \in \{1, 2\}, \quad (3.1)$$

where $h_1, h_2 \in C_0^\infty(S)$. (We stress that $G_{i,m}$ are t -dependent, although this is not reflected by the notation). Moreover, we set

$$B_m^*(G_{i,m}) := \int d^3 q d^{3m} k G_{i,m}(q; k) a^*(k)^m \eta^*(q - \underline{k}). \quad (3.2)$$

Another convention, which we will use in this and the next section, concerns contractions of creation and annihilation operators. It is explained in full detail in Lemma B.2, so it is enough to illustrate it here by a simple example. Let us set $\tilde{n} = 3, \tilde{m} = 2, n = 2, m = 3$ and consider photon variables $\tilde{r} \in \mathbb{R}^{3\tilde{n}}, \tilde{k} \in \mathbb{R}^{3\tilde{m}}, r \in \mathbb{R}^{3n}, k \in \mathbb{R}^{3m}$. Then the expectation value

$$\begin{aligned} & \langle \Omega, a(\tilde{r})^{\tilde{n}} a(\tilde{k})^{\tilde{m}} a^*(r)^n a^*(k)^m \Omega \rangle \\ &= \langle \Omega, \{a(\tilde{r}_1) a(\tilde{r}_2) a(\tilde{r}_3)\} \{a(\tilde{k}_1) a(\tilde{k}_2)\} \{a^*(r_1) a^*(r_2)\} \{a^*(k_1) a^*(k_2) a^*(k_3)\} \Omega \rangle \end{aligned} \quad (3.3)$$

is a sum of $(m+n)!$ terms resulting from all the possible contraction patterns. Let us consider one of them:

$$\langle \Omega, \{a(\tilde{r}_1) a(\tilde{r}_2) a(\tilde{r}_3)\} \{a(\tilde{k}_1) a(\tilde{k}_2)\} \{a^*(r_1) a^*(r_2)\} \{a^*(k_1) a^*(k_2) a^*(k_3)\} \Omega \rangle. \quad (3.4)$$

Given this contraction pattern, we make a decomposition $\tilde{r} = (\hat{\tilde{r}}, \check{\tilde{r}})$, where $\hat{\tilde{r}} = (\tilde{r}_1, \tilde{r}_2)$ are the \tilde{r} -variables which are contracted with some r -variables and $\check{\tilde{r}} = \tilde{r}_3$ is the \tilde{r} -variable contracted with a k -variable. In the case of the \tilde{k} -variables we have $\tilde{k} = \hat{\tilde{k}}$, since both \tilde{k}_1 and \tilde{k}_2 are contracted with k -variables. In this situation we say that $\check{\tilde{k}}$ is empty. Similarly, we have $r = \hat{r}$ with \check{r} empty and $k = (\hat{k}, \check{k})$ with $\hat{k} = (k_2, k_3)$ and $\check{k} = \{k_1\}$.

3.1 Double commutator

It turns out that the behaviour of the double commutator on the r.h.s. of (2.20) is governed by the decay of the functions

$$F_{n,m}^{G_1, G_2}(q; r | p; k) := (n+1) \int d^3 r_{n+1} v_{\alpha}(r_{n+1}) G_{1,n+1}(q + r_{n+1}; r, r_{n+1}) G_{2,m}(p - r_{n+1}; k), \quad (3.5)$$

where $q, p \in \mathbb{R}^3, r \in \mathbb{R}^{3n}, k \in \mathbb{R}^{3m}$. For any such function we define an auxiliary operator

$$B_{n,m}^*(F_{n,m}^{G_1, G_2}) := \int d^3 q d^3 p \int d^{3n} r d^{3m} k F_{n,m}^{G_1, G_2}(q; r | p; k) a^*(r)^n a^*(k)^m \eta^*(p - \underline{k}) \eta^*(q - \underline{p}). \quad (3.6)$$

Key properties of expectation values of such operators are given in Lemma B.3 below. Now we state and prove the estimate on the double commutator which gives rise to the first term on the r.h.s. of (2.4).

Proposition 3.1. *There holds the bound*

$$\|[[H_1^a, \hat{\eta}_\sigma^*(h_{1,t})], \hat{\eta}_\sigma^*(h_{2,t})]\Omega\| \leq \frac{c}{\sigma^{\delta_{\lambda_0}}} \left(\frac{\sigma^{\bar{\alpha}/(4\gamma_0)}}{t} + \sigma t + \frac{1}{t^2 \sigma^{1/(4\gamma_0)}} \right). \quad (3.7)$$

Proof. We compute

$$\begin{aligned} & \langle [[H_1^a, \hat{\eta}_\sigma^*(h_{1,t})], \hat{\eta}_\sigma^*(h_{2,t})]\Omega, [[H_1^a, \hat{\eta}_\sigma^*(h_{1,t})], \hat{\eta}_\sigma^*(h_{2,t})]\Omega \rangle \\ &= \sum_{\substack{m,n,\tilde{m},\tilde{n} \in \mathbb{N}_0 \\ m+n=\tilde{m}+\tilde{n}}} \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} \langle [[H_1^a, B_{\tilde{n}}^*(G_{1,\tilde{n}})], B_{\tilde{m}}^*(G_{2,\tilde{m}})]\Omega, [[H_1^a, B_n^*(G_{1,n})], B_m^*(G_{2,m})]\Omega \rangle, \\ &= \sum_{\substack{m,n,\tilde{m},\tilde{n} \in \mathbb{N}_0 \\ m+n=\tilde{m}+\tilde{n}}} \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} \langle (B_{\tilde{n}-1,\tilde{m}}^*(F_{\tilde{n}-1,\tilde{m}}^{G_1,G_2}) + B_{\tilde{m}-1,\tilde{n}}^*(F_{\tilde{m}-1,\tilde{n}}^{G_2,G_1}))\Omega, \\ & \quad (B_{n-1,m}^*(F_{n-1,m}^{G_1,G_2}) + B_{m-1,n}^*(F_{m-1,n}^{G_2,G_1}))\Omega \rangle, \end{aligned} \quad (3.8)$$

where in the last step we made use of Lemma 3.2 and the operators $B_{m,n}^*(\cdot)$ are defined in (3.6). Given families of functions $G_{i_1,n}, G_{i_2,n}, G_{j_1,n}, G_{j_2,n}$ of the form (3.1), with $i_1, i_2, j_1, j_2 \in \{1, 2\}$, we define

$$\begin{aligned} & C(G_{i_1}, G_{i_2}; G_{j_1}, G_{j_2}) \\ &:= \sum_{\substack{m,n,\tilde{m},\tilde{n} \in \mathbb{N}_0 \\ m+n=\tilde{m}+\tilde{n}}} \frac{1}{\sqrt{m!(n+1)!\tilde{m}!(\tilde{n}+1)!}} \langle B_{\tilde{n},\tilde{m}}^*(F_{\tilde{n},\tilde{m}}^{G_{i_1},G_{i_2}})\Omega, B_{n,m}^*(F_{n,m}^{G_{j_1},G_{j_2}})\Omega \rangle. \end{aligned} \quad (3.9)$$

Then we obtain from (3.8), taking into account that $B_{-1,m}^*(F_{-1,m}^{G_1,G_2})\Omega = B_{-1,n}^*(F_{-1,n}^{G_2,G_1})\Omega = 0$,

$$\begin{aligned} \|[[H_1^a, \hat{\eta}_\sigma^*(h_{1,t})], \hat{\eta}_\sigma^*(h_{2,t})]\Omega\|^2 &= C(G_1, G_2; G_1, G_2) + C(G_2, G_1; G_2, G_1) \\ &+ 2\text{Re}(C(G_1, G_2; G_2, G_1)). \end{aligned} \quad (3.10)$$

In view of definition (3.1), $C(G_2, G_1; G_2, G_1)$ can be obtained from $C(G_1, G_2; G_1, G_2)$ by a substitution $(h_1, h_2) \rightarrow (h_2, h_1)$. Thus it suffices to consider $C(G'_1, G'_2; G_1, G_2)$, where

$$G'_{i,m}(q; k) := e^{-iE_q t} h'_i(q) f_{q,\sigma}^m(k), \quad i \in \{1, 2\} \quad (3.11)$$

and $(h'_1, h'_2) \in \{(h_1, h_2), (h_2, h_1)\}$. We recall from Lemma B.3 that

$$\begin{aligned} & \langle B_{\tilde{n},\tilde{m}}^*(F_{\tilde{n},\tilde{m}}^{G'_1,G'_2})\Omega, B_{n,m}^*(F_{n,m}^{G_1,G_2})\Omega \rangle = \sum_{\rho \in S_{m+n}} \int d^3 q d^3 p \int d^{3n} r d^{3m} k F_{n,m}^{G_1,G_2}(q; r | p; k) \\ & \times \left(\overline{F}_{\tilde{n},\tilde{m}}^{G'_1,G'_2}(p - \hat{k} + \hat{r}; \hat{r}, \check{k} | q + \hat{k} - \hat{r}; \hat{k}, \check{r}) + \overline{F}_{\tilde{n},\tilde{m}}^{G'_1,G'_2}(q + \hat{k} - \hat{r}; \hat{r}, \check{k} | p - \hat{k} + \hat{r}; \hat{k}, \check{r}) \right). \end{aligned} \quad (3.12)$$

The notation $\hat{k}, \check{k}, \hat{r}, \check{r}$ is explained in Lemma B.2 and at the beginning of this section. Now we obtain from Lemma 4.1

$$|F_{n,m}^{G_1,G_2}(q; r | p; k)| \leq \frac{1}{\sigma^{\delta_{\lambda_0}}} \left(\frac{\sigma^{\bar{\alpha}/(4\gamma_0)}}{t} + \sigma t + \frac{1}{t^2 \sigma^{1/(4\gamma_0)}} \right) \frac{1}{\sqrt{m!n!}} D(p, q) g_\sigma^m(k) g_\sigma^n(r), \quad (3.13)$$

where $(p, q) \mapsto D(p, q)$ is a smooth, compactly supported function. From this bound we get

$$\begin{aligned} & |F_{n,m}^{G_1,G_2}(q; r | p; k) \overline{F}_{\tilde{n},\tilde{m}}^{G'_1,G'_2}(p - \hat{k} + \hat{r}; \hat{r}, \check{k} | q + \hat{k} - \hat{r}; \hat{k}, \check{r})| \\ & \leq \frac{1}{\sigma^{\delta_{\lambda_0}}} \left(\frac{\sigma^{\bar{\alpha}/(4\gamma_0)}}{t} + \sigma t + \frac{1}{t^2 \sigma^{1/(4\gamma_0)}} \right)^2 D'(p, q) \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} g_\sigma^m(k)^2 g_\sigma^n(r)^2 \end{aligned} \quad (3.14)$$

and similarly

$$\begin{aligned} & |F_{n,m}^{G_1,G_2}(q; r | p; k) \overline{F}_{\tilde{n},\tilde{m}}^{G'_1,G'_2}(q + \underline{\check{k}} - \underline{\check{r}}; \hat{r}, \check{k} | p - \underline{\check{k}} + \underline{\check{r}}; \hat{k}, \check{r})| \\ & \leq \frac{1}{\sigma^{\delta_{\lambda_0}}} \left(\frac{\sigma^{\bar{\alpha}/(4\gamma_0)}}{t} + \sigma t + \frac{1}{t^2 \sigma^{1/(4\gamma_0)}} \right)^2 D'(p, q) \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} g_\sigma^m(k)^2 g_\sigma^n(r)^2, \end{aligned} \quad (3.15)$$

where D' is again a smooth, compactly supported function. Making use of (3.9), (3.12) and the last two bounds we get

$$\begin{aligned} & |C(G'_1, G'_2; G_1, G_2)| \\ & \leq \frac{1}{\sigma^{\delta_{\lambda_0}}} \left(\frac{\sigma^{\bar{\alpha}/(4\gamma_0)}}{t} + \sigma t + \frac{1}{t^2 \sigma^{1/(4\gamma_0)}} \right)^2 \sum_{\substack{m,n,\tilde{m},\tilde{n} \in \mathbb{N}_0 \\ m+n=\tilde{m}+\tilde{n}}} \frac{(m+n)!}{\sqrt{m!(n+1)!\tilde{m}!(\tilde{n}+1)!}} \\ & \quad \times \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} \|g_\sigma^m\|_2^2 \|g_\sigma^n\|_2^2 \\ & \leq \frac{1}{\sigma^{\delta_{\lambda_0}}} \left(\frac{\sigma^{\bar{\alpha}/(4\gamma_0)}}{t} + \sigma t + \frac{1}{t^2 \sigma^{1/(4\gamma_0)}} \right)^2 \left(\frac{\kappa_*}{\sigma} \right)^{4\lambda^2 c^2} \leq \frac{c'}{\sigma^{\delta'_{\lambda_0}}} \left(\frac{\sigma^{\bar{\alpha}/(4\gamma_0)}}{t} + \sigma t + \frac{1}{t^2 \sigma^{1/(4\gamma_0)}} \right)^2, \end{aligned} \quad (3.16)$$

where we made use of Lemma 3.3 and definition (1.40). This concludes the proof. \square

Lemma 3.2. *For any $n, m \in \mathbb{N}_0$ there holds*

$$[[H_1^a, B_n^*(G_{1,n})], B_m^*(G_{2,m})]\Omega = B_{n-1,m}^*(F_{n-1,m}^{G_1,G_2})\Omega + B_{m-1,n}^*(F_{m-1,n}^{G_2,G_1})\Omega, \quad (3.17)$$

where $F_{n-1,m}^{G_1,G_2}$ is defined in (3.5) and we set $B_{-1,m}^*(F_{-1,m}^{G_1,G_2})\Omega = B_{-1,n}^*(F_{-1,n}^{G_2,G_1})\Omega = 0$.

Proof. First we compute the inner commutator on C :

$$[H_1^a, B_n^*(G_{1,n})] = \int d^3 q d^{3n} r d^3 u d^3 w G_{1,n}(q; r) v_{\bar{\alpha}}(w) [a(w) \eta^*(u + w) \eta(u), a^*(r)^n \eta^*(q - \underline{r})]. \quad (3.18)$$

We note that

$$\begin{aligned} [a(w) \eta^*(u + w) \eta(u), a^*(r)^n \eta^*(q - \underline{r})] &= \sum_{i=1}^n \delta(w - r_i) a^*(r_i)^{n-1} \eta^*(u + w) \eta(u) \eta^*(q - \underline{r}) \\ &\quad + a(w) a^*(r)^n \delta(u - q + \underline{r}) \eta^*(u + w), \end{aligned} \quad (3.19)$$

where $a^*(r_{i_*})^{n-1} = a^*(r_1) \dots a^*(r_{i-1}) a^*(r_{i+1}) \dots a^*(r_n)$ for $n \geq 1$ and $a^*(r_{i_*})^{n-1} = 0$ for $n = 0$. Since $G_{1,n}$ is symmetric in the photon variables, the contributions of the first and the second term on the r.h.s. of (3.19) are given by

$$\begin{aligned} & [H_1^a, B_n^*(G_{1,n})]_1 \\ & := n \int d^3 q d^{3(n-1)} r d^3 r_n d^3 u G_{1,n}(q; r, r_n) v_{\bar{\alpha}}(r_n) a^*(r)^{n-1} \eta^*(u + r_n) \eta(u) \eta^*(q - \underline{r} - r_n) \end{aligned} \quad (3.20)$$

and

$$[H_1^a, B_n^*(G_{2,n})]_2 := \int d^3 q d^{3n} r d^3 w G_{2,n}(q; r) v_{\bar{\alpha}}(w) a(w) a^*(r)^n \eta^*(q - \underline{r} + w), \quad (3.21)$$

respectively. Now let us compute the first contribution to the double commutator:

$$\begin{aligned}
& [[H_1^a, B_n^*(G_{1,n})]_1, B_m^*(G_{2,m})]\Omega \\
&= n \int d^3 q d^{3(n-1)} r d^3 r_n d^3 u \int d^3 p d^{3m} k G_{1,n}(q; r, r_n) v_{\vec{\alpha}}(r_n) G_{2,m}(p; k) \\
&\quad \times [a^*(r)^{n-1} \eta^*(u + r_n) \eta(u) \eta^*(q - \underline{r} - r_n), a^*(k)^m \eta^*(p - \underline{k})] \Omega \\
&= n \int d^3 q d^3 p \int d^{3(n-1)} r d^3 r_n d^{3m} k G_{1,n}(q; r, r_n) v_{\vec{\alpha}}(r_n) G_{2,m}(p; k) \\
&\quad \times a^*(r)^{n-1} a^*(k)^m \eta^*(p - \underline{k} + r_n) \eta^*(q - \underline{r} - r_n) \Omega.
\end{aligned} \tag{3.22}$$

By changing variables $p \rightarrow p - r_n$ and $q \rightarrow q + r_n$, we get

$$[[H_1^a, B_n^*(G_{1,n})]_1, B_m^*(G_{2,m})]\Omega = B_{n-1,m}^*(F_{n-1,m}^{G_1, G_2})\Omega. \tag{3.23}$$

The second contribution to the double commutator has the form:

$$\begin{aligned}
& [[H_1^a, B_n^*(G_{1,n})]_2, B_m^*(G_{2,m})]\Omega \\
&= \int d^3 q d^{3n} r d^3 w \int d^3 p d^{3m} k G_{1,n}(q; r) v_{\vec{\alpha}}(w) G_{2,m}(p; k) \\
&\quad \times [a(w) a^*(r)^n \eta^*(q - \underline{r} + w), a^*(k)^m \eta^*(p - \underline{k})] \Omega \\
&= m \int d^3 q d^3 p \int d^{3n} r d^{3(m-1)} k d^3 k_m G_{1,n}(q; r) v_{\vec{\alpha}}(k_m) G_{2,m}(p; k, k_m) \\
&\quad \times a^*(r)^n a^*(k)^{m-1} \eta^*(q - \underline{r} + k_m) \eta^*(p - \underline{k} - k_m) \Omega.
\end{aligned} \tag{3.24}$$

By changing variables $q \rightarrow q - k_m$ and $p \rightarrow p + k_m$ we obtain

$$[[H_1^a, B_n^*(G_{1,n})]_2, B_m^*(G_{2,m})]\Omega = B_{m-1,n}^*(F_{m-1,n}^{G_2, G_1})\Omega, \tag{3.25}$$

which concludes the proof. \square

Lemma 3.3. *There hold the estimates*

$$\sum_{\substack{m,n,\tilde{m},\tilde{n} \in \mathbb{N}_0, \\ m+n=\tilde{m}+\tilde{n}}} \frac{(\tilde{m} + \tilde{n})!}{m!n!\tilde{m}!\tilde{n}!} \|g_\sigma^m\|_2^2 \|g_\sigma^n\|_2^2 \leq \left(\frac{\kappa_*}{\sigma}\right)^{4\lambda^2 c^2}, \tag{3.26}$$

$$\sum_{\substack{m,n,\tilde{m},\tilde{n} \in \mathbb{N}_0, \\ m+n=\tilde{m}+\tilde{n}}} \frac{(\tilde{m} + \tilde{n})!}{m!n!\tilde{m}!\tilde{n}!} \|g_\sigma^{m-1}\|_2^2 \|g_\sigma^n\|_2^2 \leq 2 \left(\frac{\kappa_*}{\sigma}\right)^{4\lambda^2 c^2}, \tag{3.27}$$

$$\sum_{\substack{m,n,\tilde{m},\tilde{n} \in \mathbb{N}_0, \\ m+n=\tilde{m}+\tilde{n}}} \frac{(\tilde{m} + \tilde{n})!}{m!n!\tilde{m}!\tilde{n}!} \|g_\sigma^{m-1}\|_2^2 \|g_\sigma^{n-1}\|_2^2 \leq 4 \left(\frac{\kappa_*}{\sigma}\right)^{4\lambda^2 c^2}, \tag{3.28}$$

where g_σ^m, κ_* are defined in (1.40) and we set by convention $g_\sigma^{-1} = 0$.

Proof. By definition of the functions g_σ^m

$$\|g_\sigma^m\|_2^2 \leq (\lambda c)^{2m} (\log(\kappa_*/\sigma))^m. \tag{3.29}$$

Thus we get

$$\begin{aligned} \sum_{\substack{m,n,\tilde{m},\tilde{n} \in \mathbb{N}_0 \\ m+n=\tilde{m}+\tilde{n}}} \frac{(\tilde{m}+\tilde{n})!}{m!n!\tilde{m}!\tilde{n}!} \|g_\sigma^m\|_2^2 \|g_\sigma^n\|_2^2 &\leq \sum_{m,n \in \mathbb{N}_0} \left(\sum_{\substack{\tilde{m},\tilde{n} \in \mathbb{N}_0 \\ \tilde{m}+\tilde{n}=m+n}} \frac{(\tilde{m}+\tilde{n})!}{\tilde{m}!\tilde{n}!} \right) \frac{(\lambda c)^{2(m+n)}}{m!n!} (\log(\kappa_*/\sigma))^{m+n} \\ &\leq \sum_{m,n \in \mathbb{N}_0} \frac{(\sqrt{2}\lambda c)^{2(m+n)}}{m!n!} (\log(\kappa_*/\sigma))^{m+n} = \left(\frac{\kappa_*}{\sigma}\right)^{4\lambda^2 c^2}. \end{aligned} \quad (3.30)$$

Let us now prove (3.27):

$$\begin{aligned} \sum_{\substack{m,n,\tilde{m},\tilde{n} \in \mathbb{N}_0 \\ m+n=\tilde{m}+\tilde{n}}} \frac{(\tilde{m}+\tilde{n})!}{m!n!\tilde{m}!\tilde{n}!} \|g_\sigma^{m-1}\|_2^2 \|g_\sigma^n\|_2^2 &\leq \sum_{\substack{m,n,\tilde{m},\tilde{n} \in \mathbb{N}_0 \\ m+n+1=\tilde{m}+\tilde{n}}} \frac{(\tilde{m}+\tilde{n})!}{m!n!\tilde{m}!\tilde{n}!} \|g_\sigma^m\|_2^2 \|g_\sigma^n\|_2^2 \\ &\leq \sum_{m,n \in \mathbb{N}_0} \left(\sum_{\substack{\tilde{m},\tilde{n} \in \mathbb{N}_0 \\ \tilde{m}+\tilde{n}=m+n+1}} \frac{(\tilde{m}+\tilde{n})!}{\tilde{m}!\tilde{n}!} \right) \frac{(\lambda c)^{2(m+n)}}{m!n!} (\log(\kappa_*/\sigma))^{m+n} \\ &\leq 2 \sum_{m,n \in \mathbb{N}_0} \frac{(\sqrt{2}\lambda c)^{2(m+n)}}{m!n!} (\log(\kappa_*/\sigma))^{m+n} = 2 \left(\frac{\kappa_*}{\sigma}\right)^{4\lambda^2 c^2}. \end{aligned} \quad (3.31)$$

The bound (3.28) is proven analogously. This concludes the proof. \square

3.2 Clustering of scalar products

In this subsection we consider clustering properties of scalar products of scattering states approximants. We will study the expression

$$\langle \Psi'_{t,\sigma'}, \Psi_{t,\sigma} \rangle = \langle \Omega, \hat{\eta}_{\sigma'}(h'_{2,t}) \hat{\eta}_{\sigma'}(h'_{1,t}) \hat{\eta}_\sigma^*(h_{1,t}) \hat{\eta}_\sigma^*(h_{2,t}) \Omega \rangle. \quad (3.32)$$

Recalling from (1.46) that the renormalized creation operators have the form

$$\hat{\eta}_\sigma^*(h) = \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \int d^3 p d^3 k h(p) f_{p,\sigma}^m(k) a^*(k)^m \eta^*(p - \underline{k}), \quad (3.33)$$

we obtain that (3.32) is a sum of integrals over vacuum expectation values of the electron and photon creation and annihilation operators. Contractions of the electron operators give rise to two terms

$$\langle \Psi'_{t,\sigma'}, \Psi_{t,\sigma} \rangle^{(1)} := \langle \Omega, \hat{\eta}_{\sigma'}(h'_{2,t}) \hat{\eta}_{\sigma'}(h'_{1,t}) \hat{\eta}_\sigma^*(h_{1,t}) \hat{\eta}_\sigma^*(h_{2,t}) \Omega \rangle, \quad (3.34)$$

$$\langle \Psi'_{t,\sigma'}, \Psi_{t,\sigma} \rangle^{(2)} := \langle \Omega, \hat{\eta}_{\sigma'}(h'_{2,t}) \hat{\eta}_{\sigma'}(h'_{1,t}) \hat{\eta}_\sigma^*(h_{1,t}) \hat{\eta}_\sigma^*(h_{2,t}) \Omega \rangle, \quad (3.35)$$

which we call ‘direct’ and ‘exchange’, respectively. We emphasize that the contractions in (3.34) and (3.35) above do not involve the photon creation and annihilation operators. The contractions of photon variables are the subject of the remaining part of this discussion.

As for the direct term, we distinguish two types of photon contraction patterns. The first type has the form

$$\langle \Omega, a(\tilde{k})^{\tilde{m}} a(\tilde{r})^{\tilde{n}} a^*(r)^n a^*(k)^m \Omega \rangle, \quad (3.36)$$

where the monomials of creation and annihilation operators above come from the respective renormalized creation operators in (3.34). By (3.36) we mean that all the \tilde{k} -variables are contracted with k -variables and all the \tilde{r} -variables are contracted with r -variables, and that $\tilde{m} = m$, $\tilde{n} = n$. (That is, \check{k} and \check{r} are empty in the terminology introduced below equation (3.4)). Clearly, for fixed m, n there are $m!n!$ such contraction patterns and, as we will show in the proof of Proposition 3.4, after summation over m, n they give rise to the first term on the r.h.s. of (3.38), which is $\langle \psi_{h'_1, \sigma'}, \psi_{h_1, \sigma} \rangle \langle \psi_{h'_2, \sigma'}, \psi_{h_2, \sigma} \rangle$. Contraction patterns of the second type are those for which \check{k} or \check{r} are non-empty. It will be shown with the help of the non-stationary phase analysis from Lemma 4.2 that the resulting terms contribute to the rest term $R(t, \sigma, \sigma')$ on the r.h.s. of (3.38), which eventually tends to zero in the proof of Theorem 2.1.

As for the exchange term (3.35), we consider again two types of photon contraction patterns. Contractions of the first type have the form

$$\langle \Omega, a(\tilde{k})^{\tilde{m}} a(\tilde{r})^{\tilde{n}} a^*(r)^n a^*(k)^m \Omega \rangle, \quad (3.37)$$

i.e., \hat{r} and \hat{k} are empty. These contraction patterns give rise to the second term on the r.h.s. (3.38), which is $\langle \psi_{h'_1, \sigma'}, \psi_{h_2, \sigma} \rangle \langle \psi_{h'_2, \sigma'}, \psi_{h_1, \sigma} \rangle$. Contraction patterns of the second type are those for which \hat{r} or \hat{k} are non-empty. The non-stationary phase analysis from Lemma 4.3 gives that these terms contribute to $R(t, \sigma, \sigma')$.

After this overview, we are ready to state and prove the main result of this subsection which yields estimate (2.3).

Proposition 3.4. *Let $\kappa \geq \sigma' \geq \sigma > 0$ and let $\Psi_{t, \sigma}, \Psi'_{t, \sigma}$ be two scattering states approximants given by (2.1) with momentum wave functions h_1, h_2 and h'_1, h'_2 , respectively. (We recall that $\text{supp } h_1 \cap \text{supp } h_2 = \emptyset$ and $\text{supp } h'_1 \cap \text{supp } h'_2 = \emptyset$. However, $\text{supp } h_i \cap \text{supp } h'_j$, $i, j \in \{1, 2\}$ may be non-empty). Then*

$$\begin{aligned} \langle \Psi'_{t, \sigma'}, \Psi_{t, \sigma} \rangle &= \langle \psi_{h'_1, \sigma'}, \psi_{h_1, \sigma} \rangle \langle \psi_{h'_2, \sigma'}, \psi_{h_2, \sigma} \rangle \\ &\quad + \langle \psi_{h'_1, \sigma'}, \psi_{h_2, \sigma} \rangle \langle \psi_{h'_2, \sigma'}, \psi_{h_1, \sigma} \rangle + R(t, \sigma, \sigma'), \end{aligned} \quad (3.38)$$

where the rest term satisfies

$$|R(t, \sigma, \sigma')| \leq C(h, h') \frac{1}{\sigma^{\delta_{\lambda_0}}} \left(\frac{1}{t} \frac{1}{\sigma^{1/(8\gamma_0)}} + (\sigma')^{2\bar{\alpha}} \right). \quad (3.39)$$

Here $C(h, h') := c \|h_1\|_1 \|h_2\|_1 \sum_{\beta_1, \beta_2; 0 \leq |\beta_1| + |\beta_2| \leq 1} \|\partial^{\beta_1} h'_1\|_\infty \|\partial^{\beta_2} h'_2\|_\infty$ and the sum extends over multiindices β_1, β_2 .

Proof. Let us set

$$G_{i, m}(q; k) := e^{-iE_q t} h_i(q) f_{q, \sigma}^m(k), \quad (3.40)$$

$$G'_{i, m}(q; k) := e^{-iE_q t} h'_i(q) f_{q, \sigma'}^m(k), \quad (3.41)$$

for $i \in \{1, 2\}$. Now we can write

$$\langle \Psi'_{t, \sigma'}, \Psi_{t, \sigma} \rangle = \sum_{\substack{m, n, \tilde{m}, \tilde{n} \in \mathbb{N}_0 \\ \tilde{m} + \tilde{n} = m + n}} \frac{1}{\sqrt{m! n! \tilde{m}! \tilde{n}!}} \langle \Omega, B_{\tilde{n}}(G'_{1, \tilde{n}}) B_{\tilde{m}}(G'_{2, \tilde{m}}) B_n^*(G_{1, n}) B_m^*(G_{2, m}) \Omega \rangle. \quad (3.42)$$

Making use of Lemma B.4, we obtain

$$\begin{aligned}
& \langle \Omega, B_{\tilde{n}}(G'_{1,\tilde{n}})B_{\tilde{m}}(G'_{2,\tilde{m}})B_n^*(G_{1,n})B_m^*(G_{2,m})\Omega \rangle \\
&= \sum_{\rho \in \mathcal{S}_{m+n}} \int d^3 q d^3 p \int d^{3n} r d^{3m} k G_{1,n}(q; r) G_{2,m}(p; k) \\
& \times \left(\overline{G}'_{1,\tilde{n}}(q + \check{k} - \check{\ell}; \hat{r}, \check{k}) \overline{G}'_{2,\tilde{m}}(p - \check{k} + \check{\ell}; \hat{k}, \check{r}) + \overline{G}'_{1,\tilde{n}}(p - \hat{k} + \hat{\ell}; \hat{r}, \check{k}) \overline{G}'_{2,\tilde{m}}(q + \hat{k} - \hat{\ell}; \hat{k}, \check{r}) \right), \quad (3.43)
\end{aligned}$$

where the notation in (3.43) is explained in Lemma B.2.

Let us denote the summands involving the first term in the bracket on the r.h.s. of (3.43) by $I_{m,n,\tilde{m},\tilde{n}}^{(1)}$. Let $\check{I}_{m,n,\tilde{m},\tilde{n}}^{(1)}$ be such summands coming from permutations for which \check{k} or \check{r} are non-empty (cf. the discussion below formula (3.4)). We note that there are $(m+n)! - m!n!$ such permutations. Let $\langle \Psi'_{t,\sigma'}, \Psi_{t,\sigma} \rangle^{(\check{1})}$ be the contribution to (3.42) involving all the summands $\check{I}_{m,n,\tilde{m},\tilde{n}}^{(1)}$. Making use of Lemmas 4.2 and 3.3, we get

$$\begin{aligned}
|\langle \Psi'_{t,\sigma'}, \Psi_{t,\sigma} \rangle^{(\check{1})}| &\leq C(h, h') \frac{1}{\sigma^{\delta_{\lambda_0}}} \left(\frac{1}{t} \frac{1}{\sigma^{1/(8\gamma_0)}} + (\sigma')^{2\bar{\alpha}} \right) \left(\frac{\kappa_*}{\sigma} \right)^{4\lambda^2 c^2} \\
&\leq C(h, h') \frac{1}{\sigma^{\delta'_{\lambda_0}}} \left(\frac{1}{t} \frac{1}{\sigma^{1/(8\gamma_0)}} + (\sigma')^{2\bar{\alpha}} \right), \quad (3.44)
\end{aligned}$$

Clearly, this expression contributes to $R(t, \sigma, \sigma')$.

Now let $\hat{I}_{m,n,\tilde{m},\tilde{n}}^{(1)}$ denote the expressions $I_{m,n,\tilde{m},\tilde{n}}^{(1)}$ coming from permutations for which \check{k} and \check{r} are empty. (We note that there are $m!n!$ such permutations and that in this case $m = \tilde{m}$, $n = \tilde{n}$). We obtain from Lemma B.1 that

$$\begin{aligned}
\hat{I}_{m,n,\tilde{m},\tilde{n}}^{(1)} &= \int d^3 q d^{3n} r G_{1,n}(q; r) \overline{G}'_{1,\tilde{n}}(q; r) \int d^3 p d^{3m} k G_{2,m}(p; k) \overline{G}'_{2,\tilde{m}}(p; k) \\
&= \frac{1}{n!} \frac{1}{m!} \langle \Omega, B_{\tilde{n}}(G'_{1,\tilde{n}})B_n^*(G_{1,n})\Omega \rangle \langle \Omega, B_{\tilde{m}}(G'_{2,\tilde{m}})B_m^*(G_{2,m})\Omega \rangle. \quad (3.45)
\end{aligned}$$

Let $\langle \Psi'_{t,\sigma'}, \Psi_{t,\sigma} \rangle^{(\hat{1})}$ be the contribution to $\langle \Psi'_{t,\sigma'}, \Psi_{t,\sigma} \rangle$ involving all such $\hat{I}_{m,n,\tilde{m},\tilde{n}}^{(1)}$. Since the sum over permutations in (3.43) gives the compensating factor $m!n!$, we obtain

$$\begin{aligned}
\langle \Psi'_{t,\sigma'}, \Psi_{t,\sigma} \rangle^{(\hat{1})} &= \sum_{m,n,\tilde{m},\tilde{n}} \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} \langle \Omega, B_{\tilde{n}}(G'_{1,\tilde{n}})B_n^*(G_{1,n})\Omega \rangle \langle \Omega, B_{\tilde{m}}(G'_{2,\tilde{m}})B_m^*(G_{2,m})\Omega \rangle \\
&= \langle \psi_{h'_1,\sigma'}, \psi_{h_1,\sigma} \rangle \langle \psi_{h'_2,\sigma'}, \psi_{h_2,\sigma} \rangle, \quad (3.46)
\end{aligned}$$

where in the last step we compared definition (3.2) of $B_n^*(G_{i,n})$ with definition (1.46) of the renormalized creation operator and recalled that $\psi_{h_i,\sigma} = \hat{\eta}_\sigma^*(h_i)\Omega$.

The analysis of the summands involving the second term in bracket on the r.h.s. of (3.43) is analogous, so we can be brief: Let us denote such summands by $I_{m,n,\tilde{m},\tilde{n}}^{(2)}$ and let $\hat{I}_{m,n,\tilde{m},\tilde{n}}^{(2)}$ be the summands coming from permutations for which \hat{k} or \hat{r} are non-empty. We denote by $\langle \Psi'_{t,\sigma'}, \Psi_{t,\sigma} \rangle^{(\hat{2})}$ the contribution to (3.42) involving all $\hat{I}_{m,n,\tilde{m},\tilde{n}}^{(2)}$. By Lemmas 4.3 and 3.3, we get

$$|\langle \Psi'_{t,\sigma'}, \Psi_{t,\sigma} \rangle^{(\hat{2})}| \leq C(h, h') \frac{1}{\sigma^{\delta'_{\lambda_0}}} \left(\frac{1}{t} \frac{1}{\sigma^{1/(8\gamma_0)}} + (\sigma')^{2\bar{\alpha}} \right). \quad (3.47)$$

This part contributes to the rest term.

Now let $\check{I}_{m,n,\tilde{m},\tilde{n}}^{(2)}$ be the expressions $I_{m,n,\tilde{m},\tilde{n}}^{(2)}$ coming from permutations for which \hat{k} and \hat{r} are empty. Denoting the corresponding contribution to (3.42) by $\langle \Psi'_{t,\sigma'}, \Psi_{t,\sigma} \rangle^{(\check{2})}$, we get from Lemma B.1 that

$$\langle \Psi'_{t,\sigma'}, \Psi_{t,\sigma} \rangle^{(\check{2})} = \langle \psi_{h'_1,\sigma'}, \psi_{h_2,\sigma} \rangle \langle \psi_{h'_2,\sigma'}, \psi_{h_1,\sigma} \rangle. \quad (3.48)$$

This concludes the proof. \square

The above proposition has a simple corollary which gives the last term on the r.h.s. of estimate (2.4).

Corollary 3.5. *Let $h_1, h_2 \in C_0^2(S)$ have disjoint supports and let $h_2^\sigma(q) = (E_{q,\sigma} - E_q)h_2(q)$. Then*

$$\|\hat{\eta}_\sigma^*(h_{1,t})\hat{\eta}_\sigma^*(h_{2,t}^\sigma)\Omega\| \leq c\sigma^{1/2-\delta_{\lambda_0}} \left(1 + \frac{1}{t} \frac{1}{\sigma^{1/(8\gamma_0)}}\right). \quad (3.49)$$

Proof. Making use of Proposition 3.4, we get

$$\|\hat{\eta}_\sigma^*(h_{1,t})\hat{\eta}_\sigma^*(h_{2,t}^\sigma)\Omega\|^2 = \langle \psi_{h_1,\sigma}, \psi_{h_1,\sigma} \rangle \langle \psi_{h_2^\sigma,\sigma}, \psi_{h_2^\sigma,\sigma} \rangle + R(t, \sigma, \sigma), \quad (3.50)$$

where $R(t, \sigma, \sigma)$ satisfies

$$|R(t, \sigma, \sigma)| \leq C(h, h') \frac{1}{\sigma^{\delta_{\lambda_0}}} \left(\frac{1}{t} \frac{1}{\sigma^{1/(8\gamma_0)}} + (\sigma)^{2\bar{\alpha}} \right), \quad (3.51)$$

$$C(h, h) = c \|h_1\|_1 \|h_2^\sigma\|_1 \sum_{\beta_1, \beta_2; 0 \leq |\beta_1| + |\beta_2| \leq 1} \|\partial^{\beta_1} h_1\|_\infty \|\partial^{\beta_2} h_2^\sigma\|_\infty. \quad (3.52)$$

Making use of Proposition 1.1, we obtain that $C(h, h) \leq c'\sigma$ and therefore

$$|R(t, \sigma, \sigma)| \leq c\sigma^{1-\delta_{\lambda_0}} \left(1 + \frac{1}{t} \frac{1}{\sigma^{1/(8\gamma_0)}}\right). \quad (3.53)$$

Now making use of Lemma 2.3, we get $\|\psi_{h_1,\sigma}\|^2 = \|h_1\|_2^2$ and $\|\psi_{h_2^\sigma,\sigma}\|^2 = \|h_2^\sigma\|_2^2$. Exploiting Proposition 1.1 again, we get $\|\psi_{h_2^\sigma,\sigma}\|^2 \leq c\sigma^2$. Consequently, due to the constraint on the supports of h_1 and h_2 , we obtain

$$\|\hat{\eta}_\sigma^*(h_{1,t})\hat{\eta}_\sigma^*(h_{2,t}^\sigma)\Omega\|^2 \leq c'\sigma^{1-\delta_{\lambda_0}} \left(1 + \frac{1}{t} \frac{1}{\sigma^{1/(8\gamma_0)}}\right). \quad (3.54)$$

This concludes the proof. \square

3.3 Contribution involving $\check{H}_{1,\sigma}^c$

This subsection is devoted to the term involving $\check{H}_{1,\sigma}^c$ on the r.h.s. of (2.20). The following elementary proposition, which relies on Lemma B.5, gives the second term on the r.h.s. of (2.4).

Proposition 3.6. *Let $\check{H}_{1,\sigma}^c$ be defined as in (2.22). Then there holds the bound*

$$\|\hat{\eta}_\sigma^*(h_{1,t})\check{H}_{1,\sigma}^c\hat{\eta}_\sigma^*(h_{2,t})\Omega\| \leq c\sigma^{1-\delta_{\lambda_0}}. \quad (3.55)$$

Proof. We rewrite the expression from the statement of the proposition as follows:

$$\begin{aligned} & \langle \hat{\eta}_\sigma^*(h_{1,t}) \check{H}_{1,\sigma}^c \hat{\eta}_\sigma^*(h_{2,t}) \Omega, \hat{\eta}_\sigma^*(h_{1,t}) \check{H}_{1,\sigma}^c \hat{\eta}_\sigma^*(h_{2,t}) \Omega \rangle \\ &= \sum_{\substack{m,n,\tilde{m},\tilde{n} \in \mathbb{N}_0 \\ m+n=\tilde{m}+\tilde{n}}} \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} \langle \Omega, B_{\tilde{n}}(G_{2,\tilde{n}})(\check{H}_{1,\sigma}^c)^* B_{\tilde{m}}(G_{1,\tilde{m}}) B_n^*(G_{1,n}) \check{H}_{1,\sigma}^c B_m^*(G_{2,m}) \Omega \rangle. \end{aligned} \quad (3.56)$$

Now Lemma B.5 gives

$$\begin{aligned} & \langle \Omega, B_{\tilde{n}}(G_{2,\tilde{n}})(\check{H}_{1,\sigma}^c)^* B_{\tilde{m}}(G_{1,\tilde{m}}) B_n^*(G_{1,n}) \check{H}_{1,\sigma}^c B_m^*(G_{2,m}) \Omega \rangle \\ &= \sum_{\rho \in S_{m+n}} \int d^3 q d^3 p \int d^3 n r d^3 m k G_{1,n}(q; r) G_{2,m}(p; k) \\ & \quad \times \left(\int d^3 \tilde{p} \check{v}_\alpha^\sigma(\tilde{p})^2 \overline{G}_{2,\tilde{n}}(p - \tilde{p} - \hat{k} + \hat{r}; \hat{r}, \check{k}) \overline{G}_{1,\tilde{m}}(\tilde{p} + q + \hat{k} - \hat{r}; \hat{k}, \check{r}) \right. \\ & \quad \left. + \|\check{v}_\alpha^\sigma\|_2^2 \overline{G}_{2,\tilde{n}}(q + \check{k} - \check{r}; \hat{r}, \check{k}) \overline{G}_{1,\tilde{m}}(p - \check{k} + \check{r}; \hat{k}, \check{r}) \right). \end{aligned} \quad (3.57)$$

Making use of Theorem 1.2, and of definition (3.1), we obtain the bounds

$$\begin{aligned} & |G_{1,n}(q; r) G_{2,m}(p; k) \overline{G}_{2,\tilde{n}}(p - \tilde{p} - \hat{k} + \hat{r}; \hat{r}, \check{k}) \overline{G}_{1,\tilde{m}}(\tilde{p} + q + \hat{k} - \hat{r}; \hat{k}, \check{r})| \\ & \leq \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} D(p, q) g_\sigma^n(r)^2 g_\sigma^m(k)^2, \end{aligned} \quad (3.58)$$

$$\begin{aligned} & |G_{1,n}(q; r) G_{2,m}(p; k) \overline{G}_{2,\tilde{n}}(q + \check{k} - \check{r}; \hat{r}, \check{k}) \overline{G}_{1,\tilde{m}}(p - \check{k} + \check{r}; \hat{k}, \check{r})| \\ & \leq \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} D(p, q) g_\sigma^n(r)^2 g_\sigma^m(k)^2, \end{aligned} \quad (3.59)$$

where $(p, q) \mapsto D(p, q)$ is some smooth compactly supported function independent of σ, t . Consequently, the r.h.s. of (3.57) can be estimated by

$$c \|\check{v}_\alpha^\sigma\|_2^2 (m+n)! \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} \|g_\sigma^n\|_2^2 \|g_\sigma^m\|_2^2. \quad (3.60)$$

Substituting this bound to (3.56) and making use of Lemma 3.3, we get

$$\|\hat{\eta}_\sigma^*(h_{1,t}) \check{H}_{1,\sigma}^c \hat{\eta}_\sigma^*(h_{2,t}) \Omega\|^2 \leq c \|\check{v}_\alpha^\sigma\|_2^2 \left(\frac{\kappa_*}{\sigma} \right)^{4\lambda^2 c^2}. \quad (3.61)$$

Exploiting the fact that $\|\check{v}_\alpha^\sigma\|_2 \leq c\sigma$, we conclude the proof. \square

4 Non-stationary phase arguments

In this section we derive non-stationary phase estimates which entered into our analysis in Subsections 3.1 and 3.2. The spectral information from Proposition 1.1 and Theorem 1.2 is crucial for this part of our investigation.

Lemma 4.1. *Let $G_{i,m}$, $i \in \{1, 2\}$, be as specified in (3.1) and let $F_{n,m}^{G_1, G_2}$ be defined as in (3.5) i.e., it has the form*

$$F_{n,m}^{G_1, G_2}(q; r | p; k) = (n+1) \int d^3 \tilde{r} \check{v}_\alpha(\tilde{r}) e^{-i(E_{q+\tilde{r}} + E_{p-\tilde{r}})t} h_1(p - \tilde{r}) h_2(q + \tilde{r}) f_{q+\tilde{r}, \sigma}^{n+1}(r, \tilde{r}) f_{p-\tilde{r}, \sigma}^m(k), \quad (4.1)$$

where $h_1, h_2 \in C_0^2(S)$ have disjoint supports. There holds the bound

$$|F_{n,m}^{G_1, G_2}(q; r | p; k)| \leq \frac{1}{\sigma^{\delta_{\lambda_0}}} \left(\frac{\sigma^{\bar{\alpha}/(4\gamma_0)}}{t} + \sigma t + \frac{1}{t^2 \sigma^{1/(4\gamma_0)}} \right) \frac{1}{\sqrt{m!n!}} D(p, q) g_\sigma^m(k) g_\sigma^n(r), \quad (4.2)$$

where $D \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$.

Proof. Let us introduce the slow cut-off $\sigma_s := \kappa(\sigma/\kappa)^{1/(8\gamma_0)}$, which clearly satisfies $\sigma \leq \sigma_s \leq \kappa$. Let $\chi \in C^\infty(\mathbb{R}^3)$, $0 \leq \chi \leq 1$, be supported in \mathcal{B}_1 (the unit ball) and be equal to one on $\mathcal{B}_{1-\varepsilon}$ for some small $0 < \varepsilon < 1$. We set $\chi_1(\tilde{k}) = \chi(\tilde{k}/\sigma_s)$, $\chi_2(\tilde{k}) = 1 - \chi_1(\tilde{k})$ and define

$$F_{j,n,m}^{G_1, G_2}(q; r | p; k) := \int d^3 \tilde{r} v_{\bar{\alpha}}(\tilde{r}) \chi_j(\tilde{r}) e^{-i(E_{q+\tilde{r}} + E_{p-\tilde{r}})t} h_1(p - \tilde{r}) h_2(q + \tilde{r}) \times f_{q+\tilde{r}, \sigma}^{n+1}(r, \tilde{r}) f_{p-\tilde{r}, \sigma}^m(k). \quad (4.3)$$

We set $V(q, p, \tilde{r}) := E_{q+\tilde{r}} + E_{p-\tilde{r}}$ and note that by disjointness of the velocity supports of h_1, h_2 , the condition $h_1(p - \tilde{r}) h_2(q + \tilde{r}) \neq 0$, together with Proposition 1.1(a), implies that

$$|\nabla_{\tilde{r}} V(q, p, \tilde{r})| \geq \varepsilon' > 0 \quad (4.4)$$

for some fixed ε' . Thus we can write

$$e^{-iV(q, p, \tilde{r})t} = \frac{\nabla_{\tilde{r}} V(q, p, \tilde{r}) \cdot \nabla_{\tilde{r}} e^{-iV(q, p, \tilde{r})t}}{(-it)|\nabla_{\tilde{r}} V(q, p, \tilde{r})|^2}. \quad (4.5)$$

Now we define the function

$$J(q, p, \tilde{r}) := \frac{\nabla_{\tilde{r}} V(q, p, \tilde{r})}{|\nabla_{\tilde{r}} V(q, p, \tilde{r})|^2} h_1(p - \tilde{r}) h_2(q + \tilde{r}) \chi^\kappa(\tilde{r}), \quad (4.6)$$

where $\chi^\kappa \in C_0^\infty(\mathbb{R}^3)$ is equal to one on \mathcal{B}_κ and vanishes outside of a slightly larger set. We note that, by Proposition 1.1 (a) for any multiindex β s.t. $0 \leq |\beta| \leq 1$

$$|\partial_{\tilde{r}}^\beta J(q, p, \tilde{r})| \leq D(q, p), \quad (4.7)$$

where $(q, p) \mapsto D(q, p)$ is a smooth, compactly supported function. Moreover, for $0 \leq |\beta| \leq 2$

$$|\partial_{\tilde{r}}^\beta v_{\bar{\alpha}}(\tilde{r})| \leq \frac{\chi_3(\tilde{r}) |\tilde{r}|^{\bar{\alpha}}}{|\tilde{r}|^{\frac{1}{2} + |\beta|}}, \quad |\partial_{\tilde{r}}^\beta \chi_j(\tilde{r})| \leq \frac{c}{(\sigma_s)^{|\beta|}}, \quad (4.8)$$

where χ_3 is a smooth, compactly supported function, independent of σ . In addition, for $0 \leq |\beta| \leq 2$ we obtain from Theorem 1.2 (c)

$$|\partial_{\tilde{r}}^\beta f_{p-\tilde{r}, \sigma}^m(k)| \leq \frac{1}{\sqrt{m!}} \frac{c}{\sigma^{\delta_{\lambda_0}}} g_\sigma^m(k), \quad (4.9)$$

$$|\partial_{\tilde{r}}^\beta f_{q+\tilde{r}, \sigma}^{n+1}(r, \tilde{r})| \leq \frac{1}{\sqrt{n!}} \frac{c}{\sigma^{\delta_{\lambda_0}}} \frac{|\tilde{r}|^{\bar{\alpha}}}{|\tilde{r}|^{3/2 + |\beta|}} g_\sigma^n(r). \quad (4.10)$$

Now using the Gauss Law we obtain from (4.3)

$$F_{1,n,m}^{G_1, G_2}(q; r | p; k) = \frac{1}{it} \int_{|\tilde{r}| \leq \sigma_s} d^3 \tilde{r} e^{-iV(q, p, \tilde{r})t} \nabla_{\tilde{r}} \cdot \left(J(q, p, \tilde{r}) v_{\bar{\alpha}}(\tilde{r}) \chi_1(\tilde{r}) f_{q+\tilde{r}, \sigma}^{n+1}(r, \tilde{r}) f_{p-\tilde{r}, \sigma}^m(k) \right) + \frac{\sigma^2}{it} \int d\Omega(\mathbf{n}) e^{-iV(q, p, \sigma \mathbf{n})t} \mathbf{n} \cdot \left(J(q, p, \tilde{r}) v_{\bar{\alpha}}(\tilde{r}) \chi_1(\tilde{r}) f_{q+\tilde{r}, \sigma}^{n+1}(r, \tilde{r}) f_{p-\tilde{r}, \sigma}^m(k) \right) \Big|_{\tilde{r} = \sigma \mathbf{n}}, \quad (4.11)$$

where \mathbf{n} is the normal vector to the unit sphere and $d\Omega(\mathbf{n})$ is the spherical measure (that is

$$\mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \quad (4.12)$$

and $d\Omega(\mathbf{n}) = \sin \theta d\theta d\varphi$ in spherical coordinates (θ, ϕ)). Let us consider the first term on the r.h.s. of (4.11). Let I_1 be the corresponding integrand. Making use of (4.7)–(4.10), we obtain

$$\begin{aligned} |I_1| &\leq \frac{1}{\sqrt{m!n!}} \sum_{0 \leq |\beta_1| + |\beta_2| + |\beta_3| \leq 1} \frac{c}{(\sigma_s)^{|\beta_1|}} \frac{D(q, p)}{\sigma^{\delta_{\lambda_0}}} \frac{\chi_3(\tilde{r}) |\tilde{r}|^{2\bar{\alpha}}}{|\tilde{r}|^{2+|\beta_2|+|\beta_3|}} g_\sigma^n(r) g_\sigma^m(k) \\ &\leq \frac{1}{\sqrt{m!n!}} \frac{c D(q, p)}{\sigma^{\delta_{\lambda_0}}} \frac{\chi_3(\tilde{r}) |\tilde{r}|^{2\bar{\alpha}}}{|\tilde{r}|^3} g_\sigma^n(r) g_\sigma^m(k), \end{aligned} \quad (4.13)$$

where in the last step above we made use of the fact that $|\tilde{r}| \leq \sigma_s$ in the region of integration. Now let I_2 be the integrand in the boundary integral on the r.h.s. of (4.11). Making use, again, of bounds (4.7)–(4.10), we get

$$\begin{aligned} |I_2| &\leq \frac{1}{\sqrt{m!n!}} \frac{c D(q, p)}{\sigma^{\delta_{\lambda_0}}} \frac{\chi_3(\tilde{r}) |\tilde{r}|^{2\bar{\alpha}}}{|\tilde{r}|^2} g_\sigma^n(r) g_\sigma^m(k) \Big|_{\tilde{r}=\sigma \mathbf{n}} \\ &= \frac{1}{\sqrt{m!n!}} \frac{c D(q, p)}{\sigma^{\delta_{\lambda_0}}} \frac{\chi_3(\sigma \mathbf{n}) \sigma^{2\bar{\alpha}}}{\sigma^2} g_\sigma^n(r) g_\sigma^m(k). \end{aligned} \quad (4.14)$$

Thus (4.11), (4.13), (4.14) give

$$|F_{1,n,m}^{G_1,G_2}(q; r | p; k)| \leq \frac{1}{\sqrt{m!n!}} \frac{c D(q, p)}{\sigma^{\delta_{\lambda_0}}} \frac{(\sigma_s)^{2\bar{\alpha}}}{t} g_\sigma^n(r) g_\sigma^m(k), \quad (4.15)$$

which is the first contribution to the bound (4.2).

Now we consider the contribution above the slow cut-off. In this region we will have to differentiate by parts twice, so a direct application of the non-stationary phase method would result in third derivatives of the function $V(q, p, \tilde{r}) := E_{q+\tilde{r}} + E_{p-\tilde{r}}$. However, our spectral results do not include the existence of third derivatives of $p \mapsto E_p$, but rather the bound

$$|\partial_p^\beta E_{p,\sigma}| \leq \frac{c}{\sigma^{\delta_{\lambda_0}}} \quad (4.16)$$

for any multiindex β s.t. $0 \leq |\beta| \leq 3$ (cf. Proposition 1.1 (a)). To be able to exploit this bound, we introduce an auxiliary function $V_\sigma(q, p, \tilde{r}) := E_{q+\tilde{r},\sigma} + E_{p-\tilde{r},\sigma}$ and rewrite $F_{2,n,m}^{G_1,G_2}(q; r | p; k)$, defined in (4.3), as follows

$$\begin{aligned} F_{2,n,m}^{G_1,G_2}(q; r | p; k) &= \int d^3 \tilde{r} \, v_{\bar{\alpha}}(\tilde{r}) \chi_2(\tilde{r}) e^{-iV_\sigma(q,p,\tilde{r})t} h_1(p - \tilde{r}) h_2(q + \tilde{r}) \cdot f_{q+\tilde{r},\sigma}^{n+1}(r, \tilde{r}) f_{p-\tilde{r},\sigma}^m(k) \\ &\quad + R(\sigma t). \end{aligned} \quad (4.17)$$

The rest term $R(\sigma t)$ satisfies

$$\begin{aligned} |R(\sigma t)| &\leq \int d^3 \tilde{r} \, v_{\bar{\alpha}}(\tilde{r}) \chi_2(\tilde{r}) |(1 - e^{i(V(q,p,\tilde{r}) - V_\sigma(q,p,\tilde{r}))t}) h_1(p - \tilde{r}) h_2(q + \tilde{r}) f_{q+\tilde{r},\sigma}^{n+1}(r, \tilde{r}) f_{p-\tilde{r},\sigma}^m(k)| \\ &\leq c \frac{\sigma t}{\sigma^{\delta_{\lambda_0}}} g_\sigma^m(k) g_\sigma^n(r), \end{aligned} \quad (4.18)$$

where we made use of the fact that $|V(q, p, \tilde{r}) - V_\sigma(q, p, \tilde{r})| \leq c\sigma$ (Proposition 1.1, (a)) and of the bounds (4.8), (4.9), (4.10). This gives the second contribution to the bound (4.2).

Let us denote by $F_{2,\sigma,n,m}^{G_1,G_2}$ the first term on the r.h.s. of (4.17). We will estimate this term with the help of the method of non-stationary phase. Similarly as in the first part of the proof we note that by disjointness of velocity supports of h_1, h_2 , the condition $h_1(p - \tilde{r})h_2(q + \tilde{r}) \neq 0$ implies that

$$|\nabla_{\tilde{r}} V_\sigma(q, p, \tilde{r})| \geq \varepsilon'' > 0 \quad (4.19)$$

for some fixed ε'' , independent of $\sigma \in (0, \kappa]$. (Here we made use of the fact that $S \ni p \mapsto E_{p,\sigma}$ is strictly convex, uniformly in σ . Cf. Proposition 1.1 (a)). Thus we can write

$$e^{-iV_\sigma(q,p,\tilde{r})t} = \frac{\nabla_{\tilde{r}} V_\sigma(q, p, \tilde{r}) \cdot \nabla_{\tilde{r}} e^{-iV_\sigma(q,p,\tilde{r})t}}{(-it)|\nabla_{\tilde{r}} V_\sigma(q, p, \tilde{r})|^2}. \quad (4.20)$$

We define the function

$$J_\sigma(q, p, \tilde{r}) := \frac{\nabla_{\tilde{r}} V_\sigma(q, p, \tilde{r})}{|\nabla_{\tilde{r}} V_\sigma(q, p, \tilde{r})|^2} h_1(p - \tilde{r}) h_2(q + \tilde{r}) \chi^\kappa(\tilde{r}), \quad (4.21)$$

which is analogous to the function J introduced in (4.6) above. Making use of (4.16) and of (4.19), we get for all multiindices β s.t. $0 \leq |\beta| \leq 2$

$$|\partial_{\tilde{r}}^\beta J_\sigma(q, p, \tilde{r})| \leq \frac{D(q, p)}{\sigma^{\delta_{\lambda_0}}}. \quad (4.22)$$

By integrating twice by parts in the defining expression for $F_{2,\sigma,n,m}^{G_1,G_2}$, we get

$$\begin{aligned} F_{2,\sigma,n,m}^{G_1,G_2}(q; r | p; k) &= \frac{1}{(it)^2} \int d^3 \tilde{r} e^{-iV_\sigma(q,p,\tilde{r})t} \cdot \\ &\times \nabla_{\tilde{r}} \cdot \left(\frac{\nabla_{\tilde{r}} V_\sigma(q, p, \tilde{r})}{|\nabla_{\tilde{r}} V_\sigma(q, p, \tilde{r})|^2} \nabla_{\tilde{r}} \cdot (J_\sigma(q, p, \tilde{r}) v_{\tilde{r}}(\tilde{r}) \chi_2(\tilde{r}) f_{q+\tilde{r},\sigma}^{n+1}(r, \tilde{r}) f_{p-\tilde{r},\sigma}^m(k)) \right). \end{aligned} \quad (4.23)$$

In view of (4.16), (4.19), the function

$$(q, p, \tilde{r}) \mapsto \frac{\nabla_{\tilde{r}} V_\sigma(q, p, \tilde{r})}{|\nabla_{\tilde{r}} V_\sigma(q, p, \tilde{r})|^2} \quad (4.24)$$

is bounded by $c/\sigma^{\delta_{\lambda_0}}$, together with its first derivatives, on the support of $(q, p, \tilde{r}) \mapsto h_1(p - \tilde{r})h_2(q + \tilde{r})\chi^\kappa(\tilde{r})$. Thus we obtain from the bounds (4.8), (4.9), (4.10) and (4.22) that the integrand I in (4.23) satisfies

$$\begin{aligned} |I| &\leq \frac{1}{\sqrt{m!n!}} \sum_{0 \leq |\beta_1|+|\beta_2|+|\beta_3| \leq 2} \frac{c}{(\sigma_s)^{|\beta_1|}} \frac{D(q, p)}{\sigma^{\delta_{\lambda_0}}} \frac{\chi_3(\tilde{r}) |\tilde{r}|^{2\bar{\alpha}}}{|\tilde{r}|^{2+|\beta_2|+|\beta_3|}} g_\sigma^n(r) g_\sigma^m(k) \\ &\leq \frac{1}{\sqrt{m!n!}} \frac{c}{(\sigma_s)^2} \frac{D(q, p)}{\sigma^{\delta_{\lambda_0}}} \frac{\chi_3(\tilde{r}) |\tilde{r}|^{2\bar{\alpha}}}{|\tilde{r}|^2} g_\sigma^n(r) g_\sigma^m(k), \end{aligned} \quad (4.25)$$

where in the second step we made use of the fact that $(1 - \varepsilon)\sigma_s \leq |\tilde{r}|$ in the region of integration. Thus we get from (4.23) that

$$|F_{2,\sigma,n,m}^{G_1,G_2}(q; r | p; k)| \leq \frac{1}{\sqrt{m!n!}} \frac{cD(q, p)}{\sigma^{\delta_{\lambda_0}}} \frac{1}{t^2(\sigma_s)^2} g_\sigma^n(r) g_\sigma^m(k), \quad (4.26)$$

which gives the third contribution to (4.2). The factor $(n + 1)$, appearing in (4.1), can be estimated by 2^n and incorporated into the constant appearing in the definition of g_σ^n . \square

Lemma 4.2. Let $\check{I}_{m,n,\tilde{m},\tilde{n}}^{(1)}$ be defined as follows

$$\check{I}_{m,n,\tilde{m},\tilde{n}}^{(1)} := \int d^3 q d^3 p \int d^{3n} r d^{3m} k G_{1,n}(q; r) G_{2,m}(p; k) \times \overline{G}'_{1,\tilde{n}}(q + \underline{\check{k}} - \underline{\check{r}}; \hat{r}, \check{k}) \overline{G}'_{2,\tilde{m}}(p - \underline{\check{k}} + \underline{\check{r}}; \hat{k}, \check{r}), \quad (4.27)$$

where $G_{1,n}$, $G'_{2,\tilde{n}}$ appeared in (3.40), (3.41), the notation $k = (\hat{k}, \check{k})$, $r = (\hat{r}, \check{r})$ is explained in Lemma B.2 and we consider a permutation in (3.43) for which \check{k} or \check{r} are non-empty. Then there holds

$$|\check{I}_{m,n,\tilde{m},\tilde{n}}^{(1)}| \leq \left(\frac{1}{t} \left(\frac{1}{\sigma^{\delta_{\lambda_0}}} + \frac{1}{\sigma^{1/(8\gamma_0)}} \right) + (\sigma')^{2\bar{\alpha}} \right) \frac{C(h, h')}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} (\|g_\sigma^{n-1}\|_2^2 + \|g_\sigma^n\|_2^2) (\|g_\sigma^{m-1}\|_2^2 + \|g_\sigma^m\|_2^2), \quad (4.28)$$

where $C(h, h') := c \|h_1\|_1 \|h_2\|_1 \sum_{\beta_1, \beta_2; 0 \leq |\beta_1| + |\beta_2| \leq 1} \|\partial^{\beta_1} h'_1\|_\infty \|\partial^{\beta_2} h'_2\|_\infty$ and the sum extends over multiindices β_1, β_2 . We note that for \check{k} (resp. \check{r}) non-empty we have $m \neq 0$ (resp. $n \neq 0$) and there always holds $m + n = \tilde{m} + \tilde{n}$. We set by convention $g_\sigma^{-1} = 0$.

Proof. By inserting the definitions of $G_{i,n}$, $G'_{i,\tilde{n}}$, $i \in \{1, 2\}$, we obtain

$$\check{I}_{m,n,\tilde{m},\tilde{n}}^{(1)} = \int d^3 q d^3 p \int d^{3n} r d^{3m} k e^{i(E_{p-\underline{\check{k}}+\underline{\check{r}}}-E_q)t} e^{i(E_{q+\underline{\check{k}}-\underline{\check{r}}}-E_p)t} h_1(p) h_2(q) \times \overline{h}'_1(p - \underline{\check{k}} + \underline{\check{r}}) \overline{h}'_2(q + \underline{\check{k}} - \underline{\check{r}}) f_{q,\sigma}^n(r) f_{p,\sigma}^m(k) \overline{f}_{q+\underline{\check{k}}-\underline{\check{r}},\sigma'}^{\tilde{n}}(\hat{r}, \check{k}) \overline{f}_{p-\underline{\check{k}}+\underline{\check{r}},\sigma'}^{\tilde{m}}(\hat{k}, \check{r}). \quad (4.29)$$

Since the expression on the r.h.s. of (4.29) and the bound (4.28) are invariant under the substitutions $k \leftrightarrow r$, $p \leftrightarrow q$, $h_1 \leftrightarrow h_2$, $h'_1 \leftrightarrow h'_2$, $m \leftrightarrow n$, $\tilde{m} \leftrightarrow \tilde{n}$ it suffices to consider the case of non-empty \check{k} . Similarly as in the proof of Lemma 4.1, we introduce a slow infrared cut-off. Let us set $\sigma'_s := \kappa(\sigma'/\kappa)^{1/(8\gamma_0)}$, which clearly satisfies $\sigma' \leq \sigma'_s \leq \kappa$. Let $\chi \in C^\infty(\mathbb{R}^3)$, $0 \leq \chi \leq 1$, be supported in \mathcal{B}_1 (the unit ball) and be equal to one on $\mathcal{B}_{1-\varepsilon}$ for some $0 < \varepsilon < 1$. We set $\chi_1(\check{k}_1) := \chi(\check{k}_1/\sigma'_s)$, $\chi_2(\check{k}_1) := 1 - \chi_1(\check{k}_1)$, where \check{k}_1 is the first component of \check{k} , and write for $j \in \{1, 2\}$

$$\check{I}_{m,n,\tilde{m},\tilde{n}}^{(1)(j)} := \int d^3 q d^3 p \int d^{3n} r d^{3m} k e^{i(E_{p-\underline{\check{k}}+\underline{\check{r}}}-E_q)t} e^{i(E_{q+\underline{\check{k}}-\underline{\check{r}}}-E_p)t} \times h_1(p) h_2(q) \overline{h}'_1(p - \underline{\check{k}} + \underline{\check{r}}) \overline{h}'_2(q + \underline{\check{k}} - \underline{\check{r}}) f_{q,\sigma}^n(r) f_{p,\sigma}^m(k) \times (\chi_j(\check{k}_1) \overline{f}_{q+\underline{\check{k}}-\underline{\check{r}},\sigma'}^{\tilde{n}}(\hat{r}, \check{k})) \overline{f}_{p-\underline{\check{k}}+\underline{\check{r}},\sigma'}^{\tilde{m}}(\hat{k}, \check{r}). \quad (4.30)$$

Let us first consider (4.30) with $j = 1$. We conclude from Theorem 1.2 and the definition of the functions g_σ^n that

$$\begin{aligned} & |\chi_1(\check{k}_1) f_{q,\sigma}^n(r) f_{p,\sigma}^m(k) \overline{f}_{q+\underline{\check{k}}-\underline{\check{r}},\sigma'}^{\tilde{n}}(\hat{r}, \check{k}) \overline{f}_{p-\underline{\check{k}}+\underline{\check{r}},\sigma'}^{\tilde{m}}(\hat{k}, \check{r})| \\ & \leq \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} \chi_1(\check{k}_1) g_\sigma^n(r) g_\sigma^m(k) g_{\sigma'}^{\tilde{n}}(\hat{r}, \check{k}) g_{\sigma'}^{\tilde{m}}(\hat{k}, \check{r}) \\ & \leq \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} \frac{c \chi(\check{k}_1/\sigma'_s) \chi_{[\sigma', \kappa_*]}(\check{k}_1)^2 |\check{k}_1|^{2\bar{\alpha}}}{|\check{k}_1|^3} g_\sigma^n(r) g_\sigma^{m-1}(k') g_{\sigma'}^{\tilde{n}-1}(\hat{r}, \check{k}') g_{\sigma'}^{\tilde{m}}(\hat{k}, \check{r}) \\ & \leq \frac{1}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} \frac{c \chi(\check{k}_1/\sigma'_s) \chi_{[\sigma', \kappa_*]}(\check{k}_1)^2 |\check{k}_1|^{2\bar{\alpha}}}{|\check{k}_1|^3} g_\sigma^n(r)^2 g_\sigma^{m-1}(k')^2, \end{aligned} \quad (4.31)$$

where we decomposed $k = (\check{k}_1, k')$, $\check{k} = (\check{k}_1, \check{k}')$ and in the last step we made use of the fact that $g_{\sigma'}^{\tilde{m}}(\hat{k}, \check{r}) \leq g_{\sigma'}^{\tilde{m}}(\hat{k}, \check{r})$ and $g_{\sigma'}^{\tilde{n}-1}(\hat{r}, \check{k}') \leq g_{\sigma'}^{\tilde{n}-1}(\hat{r}, \check{k}')$ for $\sigma' \geq \sigma$. Substituting (4.31) to (4.30), we get

$$|\check{I}_{m,n,\tilde{m},\tilde{n}}^{(1)(2)}| \leq (\sigma'_s)^{2\tilde{\alpha}} \frac{c \|h_1\|_1 \|h_2\|_1 \|h'_1\|_\infty \|h'_2\|_\infty}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} \|g_\sigma^n\|_2^2 \|g_\sigma^{m-1}\|_2^2. \quad (4.32)$$

Let us now consider (4.30) for $j = 2$. We set $w := \check{k}' - \check{r}$ and define

$$V(\check{k}_1, p, q, w) := E_{-\check{k}_1+p-w} + E_{\check{k}_1+q+w}. \quad (4.33)$$

We note that, by disjointness of velocity supports of h_1, h_2 , the condition $h_1(-\check{k}_1+p-w)h_2(\check{k}_1+q+w) \neq 0$ implies that

$$|\nabla_{\check{k}_1} V(\check{k}_1, p, q, w)| \geq \varepsilon' > 0, \quad (4.34)$$

for some ε' independent of \check{k}_1, p, q and w . Thus we can write the following identity

$$e^{iV(\check{k}_1, p, q, w)t} = \frac{\nabla_{\check{k}_1} V(\check{k}_1, p, q, w) \cdot \nabla_{\check{k}_1} e^{iV(\check{k}_1, p, q, w)t}}{it|\nabla_{\check{k}_1} V(\check{k}_1, p, q, w)|^2}. \quad (4.35)$$

Now we define the function

$$J(\check{k}_1, p, q, w) := \frac{\nabla_{\check{k}_1} V(\check{k}_1, p, q, w)}{|\nabla_{\check{k}_1} V(\check{k}_1, p, q, w)|^2} \bar{h}'_1(-\check{k}_1+p-w) \bar{h}'_2(\check{k}_1+q+w). \quad (4.36)$$

We note that, by Proposition 1.1 (a), for any multiindex β s.t. $0 \leq |\beta| \leq 1$ there hold the bounds

$$|\partial_{\check{k}_1}^\beta J(\check{k}_1, p, q, w)| \leq c_0(h'_1, h'_2), \quad |\partial_{\check{k}_1}^\beta \chi_j(\check{k}_1)| \leq \frac{c}{(\sigma'_s)^{|\beta|}}, \quad (4.37)$$

where $c_0(h'_1, h'_2)$ has the form

$$c_0(h'_1, h'_2) = c \sum_{\beta_1, \beta_2; 0 \leq |\beta_1| + |\beta_2| \leq 1} \|\partial^{\beta_1} h'_1\|_\infty \|\partial^{\beta_2} h'_2\|_\infty, \quad (4.38)$$

where β_1, β_2 are multiindices. Moreover, we obtain from Theorem 1.2 (c)

$$|f_{q,\sigma}^n(r)| \leq \frac{1}{\sqrt{n!}} g_\sigma^n(r), \quad (4.39)$$

$$|\partial_{\check{k}_1}^\beta f_{p-\check{k}-\check{r},\sigma'}^{\tilde{m}}(\hat{k}, \check{r})| \leq \frac{1}{\sqrt{\tilde{m}!}} \left(\frac{1}{(\sigma')^{\delta_{\lambda_0}}} \right)^{|\beta|} g_{\sigma'}^{\tilde{m}}(\hat{k}, \check{r}), \quad (4.40)$$

$$|\partial_{\check{k}_1}^\beta (\chi_2(\check{k}_1) \bar{f}_{q+\check{k}-\check{r},\sigma'}^{\tilde{n}}(\hat{r}, \check{k}) f_{p,\sigma}^m(k))| \leq \frac{1}{\sqrt{\tilde{n}!m!}} \left(\frac{1}{(\sigma')^{\delta_{\lambda_0}}} + \frac{1}{\sigma'_s} \right)^{|\beta|} g_{\sigma'}^{\tilde{n}}(\hat{r}, \check{k}) g_\sigma^m(k). \quad (4.41)$$

Now coming back to formula (4.30) and integrating by parts we obtain

$$\begin{aligned} \check{I}_{m,n,\tilde{m},\tilde{n}}^{(1)(2)} &= -\frac{1}{it} \int d^3 q d^3 p \int d^3 n r d^3 m k e^{i(V(\check{k}_1, p, q, w) - V(0, p, q, 0))t} h_1(p) h_2(q) f_{q,\sigma}^n(r) \\ &\quad \times \nabla_{\check{k}_1} \cdot \left(J(\check{k}_1, p, q, w) \chi_2(\check{k}_1) \bar{f}_{q+\check{k}-\check{r},\sigma'}^{\tilde{n}}(\hat{r}, \check{k}) f_{p,\sigma}^m(k) \bar{f}_{p-\check{k}-\check{r},\sigma'}^{\tilde{m}}(\hat{k}, \check{r}) \right). \end{aligned} \quad (4.42)$$

Making use of the bounds (4.37), (4.39), (4.40), (4.41) and of the fact that $g_{\sigma'}^{\tilde{n}}(\hat{r}, \check{k}) \leq g_{\sigma}^{\tilde{n}}(\hat{r}, \check{k})$, since $\sigma' \geq \sigma$, we estimate

$$|\hat{I}_{m,n,\tilde{m},\tilde{n}}^{(1)(2)}| \leq \frac{1}{t} \left(\frac{1}{(\sigma')^{\delta_{\lambda_0}}} + \frac{1}{\sigma'_s} \right) \frac{\|h_1\|_1 \|h_2\|_1 c_0(h'_1, h'_2)}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} \|g_{\sigma}^n\|_2^2 \|g_{\sigma}^m\|_2^2. \quad (4.43)$$

Exploiting (4.43), (4.32) and the fact that $\sigma'_s = \kappa(\sigma'/\kappa)^{1/(8\gamma_0)}$, we get (4.28). Here we estimated trivially

$$\|g_{\sigma}^n\|_2^2 \|g_{\sigma}^{m-1}\|_2^2 + \|g_{\sigma}^n\|_2^2 \|g_{\sigma}^m\|_2^2 \leq (\|g_{\sigma}^{n-1}\|_2^2 + \|g_{\sigma}^n\|_2^2) (\|g_{\sigma}^{m-1}\|_2^2 + \|g_{\sigma}^m\|_2^2) \quad (4.44)$$

to obtain an expression in (4.28) which is symmetric under the substitution $m \leftrightarrow n$. \square

Lemma 4.3. *Let $\hat{I}_{m,n,\tilde{m},\tilde{n}}^{(2)}$ be defined as follows*

$$\begin{aligned} \hat{I}_{m,n,\tilde{m},\tilde{n}}^{(2)} &:= \int d^3 q d^3 p \int d^{3n} r d^{3m} k G_{1,n}(q; r) G_{2,m}(p; k) \\ &\quad \times \overline{G}'_{1,\tilde{n}}(p - \hat{k} + \hat{r}; \hat{r}, \check{k}) \overline{G}'_{2,\tilde{m}}(q + \hat{k} - \hat{r}; \hat{k}, \check{r}), \end{aligned} \quad (4.45)$$

where $G_{i,m}$, $G'_{i,m}$, $i \in \{1, 2\}$, are defined in (3.40), (3.41), the notation $k = (\hat{k}, \check{k})$, $r = (\hat{r}, \check{r})$ is explained in Lemma B.2 and we consider a permutation in (3.43) for which \hat{k} or \hat{r} are non-empty. Then there holds

$$|\hat{I}_{m,n,\tilde{m},\tilde{n}}^{(2)}| \leq \left(\frac{1}{t} \left(\frac{1}{\sigma^{\delta_{\lambda_0}}} + \frac{1}{\sigma^{1/(8\gamma_0)}} \right) + (\sigma')^{2\bar{\alpha}} \right) \frac{C(h, h')}{\sqrt{m!n!\tilde{m}!\tilde{n}!}} (\|g_{\sigma}^{n-1}\|_2^2 + \|g_{\sigma}^n\|_2^2) (\|g_{\sigma}^{m-1}\|_2^2 + \|g_{\sigma}^m\|_2^2), \quad (4.46)$$

where $C(h, h') := c \|h_1\|_1 \|h_2\|_1 \sum_{\beta_1, \beta_2; 0 \leq |\beta_1| + |\beta_2| \leq 1} \|\partial^{\beta_1} h'_1\|_{\infty} \|\partial^{\beta_2} h'_2\|_{\infty}$ and the sum extends over multiindices β_1, β_2 . We note that for \hat{k} (resp. \hat{r}) non-empty we have $m \neq 0$ (resp. $n \neq 0$) and there always holds $m + n = \tilde{m} + \tilde{n}$. We set by convention $g_{\sigma}^{-1} = 0$.

Proof. By inserting the definitions of $G_{i,m}$, $G'_{i,m}$ we obtain

$$\begin{aligned} \hat{I}_{m,n,\tilde{m},\tilde{n}}^{(2)} &= \int d^3 q d^3 p \int d^{3n} r d^{3m} k e^{i(E_{q+\hat{k}-\hat{r}} - E_q)t} e^{i(E_{p-\hat{k}+\hat{r}} - E_p)t} h_1(p) h_2(q) \\ &\quad \times \overline{h}'_1(q + \hat{k} - \hat{r}) \overline{h}'_2(p - \hat{k} + \hat{r}) f_{q,\sigma}^n(r) \overline{f}_{q+\hat{k}-\hat{r},\sigma'}^{\tilde{m}}(\hat{k}, \check{r}) f_{p,\sigma}^m(k) \overline{f}_{p-\hat{k}+\hat{r},\sigma'}^{\tilde{n}}(\hat{r}, \check{k}). \end{aligned} \quad (4.47)$$

We note that by substitutions $h'_1 \leftrightarrow h'_2$, $\tilde{m} \leftrightarrow \tilde{n}$, $\hat{k} \leftrightarrow \check{k}$, $\hat{r} \leftrightarrow \check{r}$ we obtain formula (4.29). Now the statement follows from Lemma 4.2 and the fact that (4.28) does not change under the above substitutions. \square

A Domain questions

Lemma A.1. *There exist constants $0 \leq a < 1$ and $b \geq 0$ s.t. for any $\Psi \in C^{(n)}$ there holds the bound*

$$\|H_I^{(n)} \Psi\| \leq a \|H_{\text{fr}}^{(n)} \Psi\| + b \|\Psi\|, \quad (A.1)$$

where $H_I^{(n)} = H_I|_{C^{(n)}}$, $H_{\text{fr}}^{(n)} = H_{\text{fr}}|_{C^{(n)}}$ and the constant b may depend on n .

Proof. Let us use the form of the interaction Hamiltonian appearing in formula (1.25). We have $H_I^{(n)} = H_I^{a,(n)} + H_I^{c,(n)}$, where

$$H_I^{a,(n)} := \sum_{i=1}^n \int d^3k \, v_{\vec{\alpha}}(k) e^{ikx_i} a(k) \quad (\text{A.2})$$

and $H_I^{c,(n)} = (H_I^{a,(n)})^*$. Let us set $C_n(k) := \sum_{l=1}^n e^{ikx_l}$ and compute for some $\Psi \in C^{(n)}$

$$\begin{aligned} \|H_I^{a,(n)}\Psi\| &\leq \int d^3k \, v_{\vec{\alpha}}(k) \|C_n(k)a(k)\Psi\| \leq n \int d^3k \, v_{\vec{\alpha}}(k) \|a(k)\Psi\| \\ &\leq n \|\omega^{-1/2} v_{\vec{\alpha}}\|_2 \langle \Psi, H_I \Psi \rangle^{\frac{1}{2}} \leq \frac{1}{4} \|H_I \Psi\| + n^2 \|\omega^{-1/2} v_{\vec{\alpha}}\|_2^2 \|\Psi\|, \end{aligned} \quad (\text{A.3})$$

where in the last step we anticipate that (A.1) should hold with $0 < a < 1$.

Let us now consider the creation part of $H_I^{(n)}$. Making use of the canonical commutation relations, we get

$$\begin{aligned} \|H_I^{c,(n)}\Psi\|^2 &= \int d^3k_1 d^3k_2 \, v_{\vec{\alpha}}(k_1) v_{\vec{\alpha}}(k_2) \langle C_n(k_1)^* a^*(k_1) \Psi, C_n(k_2)^* a^*(k_2) \Psi \rangle \\ &\leq \|H_I^{a,(n)}\Psi\|^2 + n^2 \|v_{\vec{\alpha}}\|_2^2 \|\Psi\|^2, \end{aligned} \quad (\text{A.4})$$

which, together with (A.3), concludes the proof. \square

Lemma A.2. *The domain \mathcal{D} , defined in (2.19), is contained in the domains of $H, H_e, H_f, H_I^{a/c}, \check{H}_{I,\sigma}$ and $\hat{\eta}^*(h)$, $h \in C_0^2(S)$. Moreover, these operators leave \mathcal{D} invariant.*

Proof. Let $\Psi_l^{r_1, r_2}$ be a vector of the form (2.17). Then

$$\begin{aligned} &\hat{\eta}_\sigma^*(h) \Psi_l^{r_1, r_2} \\ &= \sum_{m, m'} \frac{1}{\sqrt{m'}!} \frac{1}{\sqrt{m}!} \int d^3p' \, d^3l \, p \, d^{3m'} k' \, d^{3m} k \, F'_{1, m'}(p'; k') F_{l, m}(p; k) \\ &\quad \times \eta^*(p') \eta^*(p)^l a^*(k')^{m'} a^*(k)^m \Omega \\ &= \sum_{\tilde{m}=0}^{\infty} \frac{1}{\sqrt{\tilde{m}}!} \int d^{3(l+1)} \tilde{p} \, d^{3\tilde{m}} \tilde{k} \, \tilde{F}_{\tilde{m}}(\tilde{p}; \tilde{k}) \eta^*(\tilde{p})^{l+1} a^*(\tilde{k})^{\tilde{m}} \Omega, \end{aligned} \quad (\text{A.5})$$

where $F'_{1, m'}(p', k') := h(p' + \underline{k}') f_{p' + \underline{k}', \sigma}^{m'}(k')$, $\tilde{m} := m + m'$, $\tilde{k} := (k, k')$, $\tilde{p} := (p, p')$ and

$$\tilde{F}_{\tilde{m}}(\tilde{p}; \tilde{k}) = \sum_{m=0}^{\tilde{m}} \frac{\sqrt{\tilde{m}}!}{\sqrt{(\tilde{m}-m)!} \sqrt{m}!} (F'_{1, (\tilde{m}-m)} F_{l, m})_{\text{sym}}(\tilde{p}; \tilde{k}), \quad (\text{A.6})$$

where the symmetrization is performed in the \tilde{p} and \tilde{k} variables separately. By Theorem 1.2, $F'_{1, m'}$ satisfies the bound (2.18), and therefore

$$\|\tilde{F}_{\tilde{m}}\|_2 \leq \frac{c^{\tilde{m}}}{\sqrt{\tilde{m}}!}, \quad (\text{A.7})$$

for some constant c . Hence $\hat{\eta}_\sigma^*(h) \Psi_l^{r_1, r_2}$ is well defined and belongs to \mathcal{D} .

Next, we note that

$$H_e \Psi_l^{r_1, r_2} = \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \int d^3 l p d^3 m k (\Omega(p_1) + \dots + \Omega(p_l)) F_{l,m}(p; k) \eta^*(p)^l a^*(k)^m \Omega, \quad (\text{A.8})$$

$$H_f \Psi_l^{r_1, r_2} = \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \int d^3 l p d^3 m k (\omega(k_1) + \dots + \omega(k_m)) F_{l,m}(p; k) \eta^*(p)^l a^*(k)^m \Omega. \quad (\text{A.9})$$

Due to the support properties of $F_{l,m}(p; k)$ these vectors are well defined and belong to \mathcal{D} .

Finally, we consider the operators $H_I^{a/c}$. We recall that the interaction Hamiltonian restricted to $\mathcal{H}^{(l)}$ has the form

$$H_I^{(l)} := \sum_{i=1}^l \int d^3 k v_{\vec{\alpha}}(k) (e^{ikx_i} a(k) + e^{-ikx_i} a^*(k)) \quad (\text{A.10})$$

and we express $\Psi_l^{r_1, r_2}$ in terms of its m -photon components, i.e.,

$$\{\Psi_l^{r_1, r_2}\}^{(l,m)}(p_1, \dots, p_l; k_1, \dots, k_m) = F_{l,m}(p_1, \dots, p_l; k_1, \dots, k_m), \quad (\text{A.11})$$

Now we can write

$$\begin{aligned} & (H_I^a \Psi_l^{r_1, r_2})^{(l,m)}(p_1, \dots, p_l; k_1, \dots, k_m) \\ &= \sqrt{m+1} \int d^3 k v_{\vec{\alpha}}(k) \sum_{i=1}^l (\Psi_l^{r_1, r_2})^{(l,m+1)}(p_1, \dots, p_i - k, \dots, p_l; k, k_1, \dots, k_m). \end{aligned} \quad (\text{A.12})$$

It is easy to see that for some constant c , independent of m ,

$$\|(H_I^a \Psi_l^{r_1, r_2})^{(l,m)}\|_2 \leq \frac{c^m}{\sqrt{m!}}. \quad (\text{A.13})$$

Similarly, we obtain that

$$\begin{aligned} & (H_I^c \Psi_l^{r_1, r_2})^{(l,m)}(p_1, \dots, p_l; k_1, \dots, k_m) \\ &= \frac{1}{\sqrt{m}} \sum_{i=1}^m \sum_{j=1}^l v_{\vec{\alpha}}(k_i) (\Psi_l^{r_1, r_2})^{(l,m-1)}(p_1, \dots, p_j + k_i, \dots, p_l; k_1, \dots, k_{i_*}, \dots, k_m), \end{aligned} \quad (\text{A.14})$$

where k_{i_*} means omission of the i -th variable. This gives, again, a bound of the form (A.13). Since the case of $\check{H}_{I,\sigma}$ is analogous, this concludes the proof. \square

B Fock space combinatorics

In Lemmas B.1, B.3 and B.5 below we verify the identities first for $G_{i,m}, G'_{i,m}, F_{n,m}, F'_{n,m}$ of Schwartz class and then extend them to square integrable functions using Theorem X.44 of [RS2]

Lemma B.1. *Let $G_m, G'_m \in L^2(\mathbb{R}^3 \times \mathbb{R}^{3m})$ be symmetric in their photon variables, see (1.61). Let us define as operators on \mathcal{C}*

$$B_m^*(G_m) := \int d^3 p d^3 m k G_m(p; k) \eta^*(p - \underline{k}) a^*(k)^m \quad (\text{B.1})$$

and $B_m(G_m) := (B_m^*(G_m))^*$. Then there holds the identity

$$\langle \Omega, B_m(G'_m) B_m^*(G_m) \Omega \rangle = m! \int d^3 p d^{3m} k \overline{G'_m}(p; k) G_m(p; k). \quad (\text{B.2})$$

Proof. We compute

$$\begin{aligned} & \langle \Omega, B_m(G'_m) B_m^*(G_m) \Omega \rangle \\ &= \int d^3 p d^{3m} k \int d^3 p' d^{3m} k' G_m(p; k) \overline{G'_m}(p'; k') \langle \Omega, a(k')^m \eta(p' - \underline{k}') \eta^*(p - \underline{k}) a^*(k)^m \Omega \rangle \\ &= \int d^3 p d^{3m} k \int d^3 p' d^{3m} k' G_m(p; k) \overline{G'_m}(p'; k') \delta(p - \underline{k} - p' + \underline{k}') \langle \Omega, a(k')^m a^*(k)^m \Omega \rangle \\ &= \int d^3 p d^{3m} k \int d^{3m} k' G_m(p; k) \overline{G'_m}(p - \underline{k} + \underline{k}'; k') \langle \Omega, a(k')^m a^*(k)^m \Omega \rangle \\ &= \int d^3 p d^{3m} k \int d^{3m} k' G_m(p; k) \overline{G'_m}(p - \underline{k} + \underline{k}'; k') \sum_{\rho \in S_m} \prod_{i=1}^m \delta(k_{\rho(i)} - k'_i) \\ &= m! \int d^3 p d^{3m} k G_m(p; k) \overline{G'_m}(p; k), \end{aligned} \quad (\text{B.3})$$

where S_m is the set of all permutations of an m -element set and in the last step we exploited the fact that G'_m is symmetric in its photon variables. \square

Lemma B.2. Let $n, m, \tilde{n}, \tilde{m} \in \mathbb{N}_0$ be s.t. $n + m = \tilde{n} + \tilde{m}$. Let us choose

$$r = (r_1, \dots, r_n) \in \mathbb{R}^{3n}, \quad k = (k_1, \dots, k_m) \in \mathbb{R}^{3m}, \quad (\text{B.4})$$

$$\tilde{r} = (\tilde{r}_1, \dots, \tilde{r}_{\tilde{n}}) \in \mathbb{R}^{3\tilde{n}}, \quad \tilde{k} = (\tilde{k}_1, \dots, \tilde{k}_{\tilde{m}}) \in \mathbb{R}^{3\tilde{m}} \quad (\text{B.5})$$

and define the sets

$$C_n := \{1, \dots, n\}, \quad C'_n := \{n+1, \dots, n+m\}, \quad (\text{B.6})$$

$$C_{\tilde{n}} := \{1, \dots, \tilde{n}\}, \quad C'_{\tilde{n}} := \{\tilde{n}+1, \dots, \tilde{n}+\tilde{m}\}. \quad (\text{B.7})$$

(Note that C'_n is the complement of C_n in $\{1, \dots, n+m\}$. Similarly for $C'_{\tilde{n}}$). Let S_{m+n} be the set of permutations of an $m+n$ element set. For any $\rho \in S_{m+n}$ we introduce the following notation:

$$\hat{r} := (r_i)_{(i, \rho(i)) \in C_n \times C_{\tilde{n}}}, \quad \check{r} := (r_i)_{(i, \rho(i)) \in C_n \times C'_{\tilde{n}}}, \quad (\text{B.8})$$

$$\hat{k} := (k_{i-n})_{(i, \rho(i)) \in C'_n \times C'_{\tilde{n}}}, \quad \check{k} := (k_{i-n})_{(i, \rho(i)) \in C'_n \times C_{\tilde{n}}}, \quad (\text{B.9})$$

so that $r = (\hat{r}, \check{r})$, $k = (\hat{k}, \check{k})$. Similarly,

$$\hat{\tilde{r}} := (\tilde{r}_{\rho(i)})_{(i, \rho(i)) \in C_n \times C_{\tilde{n}}}, \quad \check{\tilde{r}} := (\tilde{r}_{\rho(i)})_{(i, \rho(i)) \in C'_n \times C_{\tilde{n}}}, \quad (\text{B.10})$$

$$\hat{\tilde{k}} := (\tilde{k}_{\rho(i)-\tilde{n}})_{(i, \rho(i)) \in C'_n \times C'_{\tilde{n}}}, \quad \check{\tilde{k}} := (\tilde{k}_{\rho(i)-\tilde{n}})_{(i, \rho(i)) \in C_n \times C'_{\tilde{n}}}, \quad (\text{B.11})$$

so that $\tilde{r} = (\hat{\tilde{r}}, \check{\tilde{r}})$ and $\tilde{k} = (\hat{\tilde{k}}, \check{\tilde{k}})$. (If $\{i \mid (i, \rho(i)) \in C_n \times C_{\tilde{n}}\} = \emptyset$ then we say that \hat{r} is empty, and

analogously for other collections of photon variables introduced above). Finally, we define

$$\delta(\hat{r} - \hat{r}) := \prod_{(i, \rho(i)) \in C_n \times C_{\tilde{n}}} \delta(r_i - \tilde{r}_{\rho(i)}), \quad (\text{B.12})$$

$$\delta(\check{r} - \check{k}) := \prod_{(i, \rho(i)) \in C_n \times C'_{\tilde{n}}} \delta(r_i - \tilde{k}_{\rho(i) - \tilde{n}}), \quad (\text{B.13})$$

$$\delta(\check{k} - \check{r}) := \prod_{(i, \rho(i)) \in C'_n \times C_{\tilde{n}}} \delta(k_{i-n} - \tilde{r}_{\rho(i)}), \quad (\text{B.14})$$

$$\delta(\hat{k} - \hat{k}) := \prod_{(i, \rho(i)) \in C'_n \times C'_{\tilde{n}}} \delta(k_{i-n} - \tilde{k}_{\rho(i) - \tilde{n}}). \quad (\text{B.15})$$

Then there holds

$$\langle \Omega, a(\tilde{r})^{\tilde{n}} a(\check{k})^{\tilde{m}} a^*(r)^n a^*(k)^m \Omega \rangle = \sum_{\rho \in S_{m+n}} \delta(\hat{r} - \hat{r}) \delta(\check{r} - \check{k}) \delta(\check{k} - \check{r}) \delta(\hat{k} - \hat{k}). \quad (\text{B.16})$$

Proof. Let $(v_1, \dots, v_{n+m}) = (r_1, \dots, r_n, k_1, \dots, k_m)$ and $(\tilde{v}_1, \dots, \tilde{v}_{n+m}) = (\tilde{r}_1, \dots, \tilde{r}_{\tilde{n}}, \tilde{k}_1, \dots, \tilde{k}_{\tilde{m}})$. There holds

$$\begin{aligned} \langle \Omega, a(\tilde{r})^{\tilde{n}} a(\check{k})^{\tilde{m}} a^*(r)^n a^*(k)^m \Omega \rangle &= \sum_{\rho \in S_{m+n}} \prod_{j=1}^{m+n} \delta(v_j - \tilde{v}_{\rho(j)}) \\ &= \sum_{\rho \in S_{m+n}} \delta(r_1 - \tilde{v}_{\rho(1)}) \dots \delta(r_n - \tilde{v}_{\rho(n)}) \delta(k_1 - \tilde{v}_{\rho(n+1)}) \dots \delta(k_m - \tilde{v}_{\rho(n+m)}) \end{aligned} \quad (\text{B.17})$$

$$\begin{aligned} &= \sum_{\rho \in S_{m+n}} \left(\prod_{(i, \rho(i)) \in C_n \times C_{\tilde{n}}} \delta(r_i - \tilde{r}_{\rho(i)}) \right) \left(\prod_{(i, \rho(i)) \in C_n \times C'_{\tilde{n}}} \delta(r_i - \tilde{k}_{\rho(i) - \tilde{n}}) \right) \\ &\quad \times \left(\prod_{(i, \rho(i)) \in C'_n \times C_{\tilde{n}}} \delta(k_{i-n} - \tilde{r}_{\rho(i)}) \right) \left(\prod_{(i, \rho(i)) \in C'_n \times C'_{\tilde{n}}} \delta(k_{i-n} - \tilde{k}_{\rho(i) - \tilde{n}}) \right), \end{aligned} \quad (\text{B.18})$$

which concludes the proof. \square

Lemma B.3. Let $F_{n,m}, F'_{n,m} \in L^2((\mathbb{R}^3 \times \mathbb{R}^{3n}) \times (\mathbb{R}^3 \times \mathbb{R}^{3m}))$ be symmetric in the photon variables. Let us introduce the following operators on C

$$B_{n,m}^*(F_{n,m}) := \int d^3 q d^3 p \int d^{3n} r d^{3m} k F_{n,m}(q; r | p; k) a^*(r)^n a^*(k)^m \eta^*(p - \underline{k}) \eta^*(q - \underline{r}) \quad (\text{B.19})$$

and set $B_{n,m}(F_{n,m}) := (B_{n,m}^*(F_{n,m}))^*$. There holds

$$\begin{aligned} \langle B_{\tilde{n}, \tilde{m}}(F'_{\tilde{n}, \tilde{m}})^* \Omega, B_{n,m}(F_{n,m})^* \Omega \rangle &= \sum_{\rho \in S_{m+n}} \int d^3 q d^3 p \int d^{3n} r d^{3m} k F_{n,m}(q; r | p; k) \\ &\quad \times \left(\overline{F'}_{\tilde{n}, \tilde{m}}(p - \hat{k} + \hat{r}; \hat{r}, \check{k} | q + \hat{k} - \hat{r}; \hat{k}, \check{r}) + \overline{F'}_{\tilde{n}, \tilde{m}}(q + \check{k} - \check{r}; \hat{r}, \check{k} | p - \check{k} + \check{r}; \hat{k}, \check{r}) \right) \end{aligned} \quad (\text{B.20})$$

for $n + m = \tilde{n} + \tilde{m}$. Otherwise the expression on the l.h.s. is zero. Here S_{m+n} is the set of permutations of an $m + n$ element set and the notation $\hat{k}, \check{k}, \hat{r}, \check{r}$ is explained in Lemma B.2.

Proof. We compute the expectation value

$$\langle B_{\tilde{n},\tilde{m}}^*(F'_{\tilde{n},\tilde{m}})\Omega, B_{n,m}^*(F_{n,m})\Omega \rangle \quad (\text{B.21})$$

$$\begin{aligned} &= \int d^3\tilde{q}d^3\tilde{p}d^3qd^3p \int d^{3\tilde{n}}\tilde{r}d^{3\tilde{m}}\tilde{k}d^{3n}rd^{3m}k \overline{F}'_{\tilde{n},\tilde{m}}(\tilde{q};\tilde{r}|\tilde{p};\tilde{k})F_{n,m}(q;r|p;k) \\ &\quad \times (\delta(\tilde{q}-p+\tilde{k}-\underline{r}) + \delta(\tilde{q}-q-\tilde{r}+\underline{r}))\delta(\tilde{p}+\tilde{q}-p-q) \\ &\quad \times \langle \Omega, a(\tilde{r})^{\tilde{n}}a(\tilde{k})^{\tilde{m}}a^*(r)^na^*(k)^m\Omega \rangle. \end{aligned} \quad (\text{B.22})$$

The last factor is non-zero only if $\tilde{n} + \tilde{m} = n + m$. Then

$$\langle \Omega, a(\tilde{r})^{\tilde{n}}a(\tilde{k})^{\tilde{m}}a^*(r)^na^*(k)^m\Omega \rangle = \sum_{\rho \in S_{m+n}} \delta(\hat{r}-\hat{\tilde{r}})\delta(\check{r}-\check{\tilde{r}})\delta(\hat{k}-\hat{\tilde{k}})\delta(\check{k}-\check{\tilde{k}}), \quad (\text{B.23})$$

where we made use of Lemma B.2. Thus the r.h.s. of (B.22) is a sum over $\rho \in S_{m+n}$ of terms of the form:

$$\begin{aligned} &\int d^3\tilde{q}d^3\tilde{p}d^3qd^3p \int d^{3n}rd^{3m}k \overline{F}'_{\tilde{n},\tilde{m}}(\tilde{q};\hat{r},\check{k}|\tilde{p};\hat{k},\check{r})F_{n,m}(q;r|p;k) \\ &\quad \times \delta(\tilde{p}+\tilde{q}-p-q)(\delta(\tilde{q}-p+\hat{k}-\hat{\tilde{r}}) + \delta(\tilde{q}-q-\check{k}+\check{\tilde{r}})) \\ &= \int d^3qd^3p \int d^{3n}rd^{3m}k F_{n,m}(q;r|p;k) \left(\overline{F}'_{\tilde{n},\tilde{m}}(p-\hat{k}+\hat{\tilde{r}};\hat{r},\check{k}|q+\hat{k}-\hat{\tilde{r}};\hat{k},\check{r}) \right. \\ &\quad \left. + \overline{F}'_{\tilde{n},\tilde{m}}(q+\check{k}-\check{\tilde{r}};\hat{r},\check{k}|p-\check{k}+\check{\tilde{r}};\hat{k},\check{r}) \right) \end{aligned} \quad (\text{B.24})$$

which concludes the proof. \square

Lemma B.4. Let $G_{1,m}, G'_{1,m}, G_{2,m}, G'_{2,m} \in L^2(\mathbb{R}^3 \times \mathbb{R}^{3m})$ be symmetric in the photon variables. We define, as an operator on \mathcal{C} ,

$$B_m^*(G_{i,m}) := \int d^3p d^{3m}k G_{i,m}(p;k)\eta^*(p-\underline{k})a^*(k)^m \quad (\text{B.25})$$

and set $B_m(G_{i,m}) = (B_m^*(G_{i,m}))^*$. There holds the identity

$$\begin{aligned} &\langle \Omega, B_{\tilde{n}}(G'_{1,\tilde{n}})B_{\tilde{m}}(G'_{2,\tilde{m}})B_n^*(G_{1,n})B_m^*(G_{2,m})\Omega \rangle \\ &= \sum_{\rho \in S_{m+n}} \int d^3qd^3p \int d^{3n}rd^{3m}k G_{1,n}(q;r)G_{2,m}(p;k) \\ &\quad \times \left(\overline{G}'_{1,\tilde{n}}(p-\hat{k}+\hat{\tilde{r}};\hat{r},\check{k})\overline{G}'_{2,\tilde{m}}(q+\hat{k}-\hat{\tilde{r}};\hat{k},\check{r}) + \overline{G}'_{1,\tilde{n}}(q+\check{k}-\check{\tilde{r}};\hat{r},\check{k})\overline{G}'_{2,\tilde{m}}(p-\check{k}+\check{\tilde{r}};\hat{k},\check{r}) \right), \end{aligned} \quad (\text{B.26})$$

for $n + m = \tilde{n} + \tilde{m}$. Otherwise the expression on the l.h.s. is zero. Here S_{m+n} is the set of permutations of an $m + n$ element set and the notation $\hat{k}, \check{k}, \hat{r}, \check{r}$ is explained in Lemma B.2.

Proof. Follows immediately from Lemma B.3. \square

Lemma B.5. Let $G_{1,m}, G_{2,m} \in L^2(\mathbb{R}^3 \times \mathbb{R}^{3m})$ be supported in $\mathbb{R}^3 \times \{k \in \mathbb{R}^3 \mid |k| \geq \sigma\}^{\times m}$ and symmetric in their photon variables. There holds the identity

$$\begin{aligned} &\langle \Omega, B_{\tilde{n}}(G_{2,\tilde{n}})(\check{H}_{1,\sigma}^c)^*B_{\tilde{m}}(G_{1,\tilde{m}})B_n^*(G_{1,n})\check{H}_{1,\sigma}^cB_m^*(G_{2,m})\Omega \rangle \\ &= \sum_{\rho \in S_{m+n}} \int d^3qd^3p \int d^{3n}rd^{3m}k G_{1,n}(q;r)G_{2,m}(p;k) \\ &\quad \times \left(\int d^3\tilde{p} \check{v}_{\alpha}^{\sigma}(\tilde{p})^2 \overline{G}_{2,\tilde{n}}(p-\tilde{p}-\hat{k}+\hat{\tilde{r}};\hat{r},\check{k})\overline{G}_{1,\tilde{m}}(\tilde{p}+q+\hat{k}-\hat{\tilde{r}};\hat{k},\check{r}) \right. \\ &\quad \left. + \|\check{v}_{\alpha}^{\sigma}\|_2^2 \overline{G}_{2,\tilde{n}}(q+\check{k}-\check{\tilde{r}};\hat{r},\check{k})\overline{G}_{1,\tilde{m}}(p-\check{k}+\check{\tilde{r}};\hat{k},\check{r}) \right) \end{aligned} \quad (\text{B.27})$$

for $m+n = \tilde{m} + \tilde{n}$, otherwise the l.h.s. is zero. Here S_{m+n} is the set of permutations of an $m+n$ element set and the notation $\hat{k}, \check{k}, \hat{r}, \check{r}$ is explained in Lemma B.2.

Proof. We compute the expectation value

$$\begin{aligned}
& \langle \Omega, B_{\tilde{n}}(G_{2,\tilde{n}})(\check{H}_{1,\sigma}^c)^* B_{\tilde{m}}(G_{1,\tilde{m}}) B_n^*(G_{1,n}) \check{H}_{1,\sigma}^c B_m^*(G_{2,m}) \Omega \rangle \\
&= \int d^3 \tilde{u} d^3 \tilde{w} d^3 u d^3 w \int d^3 \tilde{q} d^3 \tilde{p} d^3 q d^3 p \int d^{3\tilde{n}} \tilde{r} d^{3\tilde{m}} \tilde{k} d^{3n} r d^{3m} k \\
& \quad \check{v}_\alpha^\sigma(\tilde{w}) \check{v}_\alpha^\sigma(w) \overline{G}_{2,\tilde{n}}(\tilde{q}; \tilde{r}) \overline{G}_{1,\tilde{m}}(\tilde{p}; \tilde{k}) G_{1,n}(q; r) G_{2,m}(p; k) \\
& \quad \times \langle \Omega, \eta(\tilde{p} - \tilde{k}) \eta^*(\tilde{u}) \eta(\tilde{u} - \tilde{w}) \eta(\tilde{q} - \tilde{r}) \eta^*(q - r) \eta^*(u - w) \eta(u) \eta^*(p - k) \Omega \rangle \\
& \quad \times \langle \Omega, a(\tilde{r})^{\tilde{n}} a(\tilde{w}) a(\tilde{k})^{\tilde{m}} a^*(r)^n a^*(w) a^*(k)^m \Omega \rangle. \tag{B.28}
\end{aligned}$$

We note that

$$\begin{aligned}
& \langle \Omega, \eta(\tilde{p} - \tilde{k}) \eta^*(\tilde{u}) \eta(\tilde{u} - \tilde{w}) \eta(\tilde{q} - \tilde{r}) \eta^*(q - r) \eta^*(u - w) \eta(u) \eta^*(p - k) \Omega \rangle \\
&= \delta(\tilde{p} - \tilde{k} - \tilde{u}) \delta(p - \underline{k} - u) \langle \Omega, \eta(\tilde{u} - \tilde{w}) \eta(\tilde{q} - \tilde{r}) \eta^*(q - r) \eta^*(u - w) \Omega \rangle \\
&= \delta(\tilde{p} - \tilde{k} - \tilde{u}) \delta(p - \underline{k} - u) \left(\delta(\tilde{u} - \tilde{w} - q + r) \delta(\tilde{q} - \tilde{r} - u + w) \right. \\
& \quad \left. + \delta(\tilde{u} - \tilde{w} - u + w) \delta(\tilde{q} - \tilde{r} - q + r) \right). \tag{B.29}
\end{aligned}$$

Let us consider the contribution to (B.28) of the first term in the bracket in (B.29):

$$\begin{aligned}
& \langle \Omega, B_{\tilde{n}}(G_{2,\tilde{n}})(\check{H}_{1,\sigma}^c)^* B_{\tilde{m}}(G_{1,\tilde{m}}) B_n^*(G_{1,n}) \check{H}_{1,\sigma}^c B_m^*(G_{2,m}) \Omega \rangle_1 \\
&:= \int d^3 \tilde{u} d^3 \tilde{w} d^3 u d^3 w \int d^3 \tilde{q} d^3 \tilde{p} d^3 q d^3 p \int d^{3\tilde{n}} \tilde{r} d^{3\tilde{m}} \tilde{k} d^{3n} r d^{3m} k \\
& \quad \check{v}_\alpha^\sigma(\tilde{w}) \check{v}_\alpha^\sigma(w) \overline{G}_{2,\tilde{n}}(\tilde{q}; \tilde{r}) \overline{G}_{1,\tilde{m}}(\tilde{p}; \tilde{k}) G_{1,n}(q; r) G_{2,m}(p; k) \\
& \quad \times \delta(\tilde{p} - \tilde{k} - \tilde{u}) \delta(p - \underline{k} - u) \delta(\tilde{u} - \tilde{w} - q + r) \delta(\tilde{q} - \tilde{r} - u + w) \\
& \quad \times \langle \Omega, a(\tilde{r})^{\tilde{n}} a(\tilde{w}) a(\tilde{k})^{\tilde{m}} a^*(r)^n a^*(w) a^*(k)^m \Omega \rangle \\
&= \int d^3 \tilde{q} d^3 \tilde{p} d^3 q d^3 p \int d^{3\tilde{n}} \tilde{r} d^{3\tilde{m}} \tilde{k} d^{3n} r d^{3m} k \\
& \quad \check{v}_\alpha^\sigma(\tilde{w}_*) \check{v}_\alpha^\sigma(w_*) \overline{G}_{2,\tilde{n}}(\tilde{q}; \tilde{r}) \overline{G}_{1,\tilde{m}}(\tilde{p}; \tilde{k}) G_{1,n}(q; r) G_{2,m}(p; k) \\
& \quad \times \langle \Omega, a(\tilde{r})^{\tilde{n}} a(\tilde{k})^{\tilde{m}} a(\tilde{w}_*) a^*(w_*) a^*(r)^n a^*(k)^m \Omega \rangle, \tag{B.30}
\end{aligned}$$

where in the last step we integrated over $u, w, \tilde{u}, \tilde{w}$ and set $w_* := p - \underline{k} - \tilde{q} + \tilde{r}$, $\tilde{w}_* := \tilde{p} - \tilde{k} - q + \underline{r}$. Now we consider the expectation value of the photon creation operators:

$$\langle \Omega, a(\tilde{r})^{\tilde{n}} a(\tilde{k})^{\tilde{m}} a(\tilde{w}_*) a^*(w_*) a^*(r)^n a^*(k)^m \Omega \rangle = \delta(w_* - \tilde{w}_*) \langle \Omega, a(\tilde{r})^{\tilde{n}} a(\tilde{k})^{\tilde{m}} a^*(r)^n a^*(k)^m \Omega \rangle, \tag{B.31}$$

for $r, k, w, \tilde{r}, \tilde{k}, \tilde{w}$ in the supports of the respective functions. (Here we made use of the fact that $|\tilde{w}_*| \leq \sigma$, whereas $|r_i| \geq \sigma, |k_j| \geq \sigma$). Let us now substitute the r.h.s. of (B.31) to (B.30). Making use of Lemma B.2, we obtain

$$\begin{aligned}
& \langle \Omega, B_{\tilde{n}}(G_{2,\tilde{n}})(\check{H}_{1,\sigma}^c)^* B_{\tilde{m}}(G_{1,\tilde{m}}) B_n^*(G_{1,n}) \check{H}_{1,\sigma}^c B_m^*(G_{2,m}) \Omega \rangle_1 \\
&= \sum_{\rho \in S_{m+n}} \int d^3 \tilde{q} d^3 \tilde{p} d^3 q d^3 p \int d^{3\tilde{n}} \tilde{r} d^{3\tilde{m}} \tilde{k} d^{3n} r d^{3m} k \\
& \quad \check{v}_\alpha^\sigma(\tilde{p} - \tilde{k} - q + \underline{r}) \check{v}_\alpha^\sigma(p - \underline{k} - \tilde{q} + \tilde{r}) \overline{G}_{2,\tilde{n}}(\tilde{q}; \tilde{r}) \overline{G}_{1,\tilde{m}}(\tilde{p}; \tilde{k}) G_{1,n}(q; r) G_{2,m}(p; k) \\
& \quad \times \delta(p + q - \tilde{p} - \tilde{q}) \delta(\hat{r} - \hat{r}) \delta(\check{r} - \check{k}) \delta(\hat{k} - \hat{k}) \delta(\check{k} - \check{r}). \tag{B.32}
\end{aligned}$$

By integrating over $\tilde{q}, \tilde{r}, \tilde{k}$, we obtain

$$\begin{aligned}
& \langle \Omega, B_{\tilde{n}}(G_{2,\tilde{n}})(\check{H}_{1,\sigma}^c)^* B_{\tilde{m}}(G_{1,\tilde{m}}) B_n^*(G_{1,n}) \check{H}_{1,\sigma}^c B_m^*(G_{2,m}) \Omega \rangle_1 \\
&= \sum_{\rho \in \mathcal{S}_{m+n}} \int d^3 \tilde{p} d^3 q d^3 p \int d^{3n} r d^{3m} k \check{v}_{\alpha}^{\sigma}(\tilde{p} - q - \hat{k} + \hat{r})^2 \\
&\quad \times \overline{G}_{2,\tilde{n}}(p + q - \tilde{p}; \hat{r}, \check{k}) \overline{G}_{1,\tilde{m}}(\tilde{p}; \hat{k}, \check{r}) G_{1,n}(q; r) G_{2,m}(p; k) \\
&= \sum_{\rho \in \mathcal{S}_{m+n}} \int d^3 \tilde{p} d^3 q d^3 p \int d^{3n} r d^{3m} k \check{v}_{\alpha}^{\sigma}(\tilde{p})^2 \\
&\quad \times \overline{G}_{2,\tilde{n}}(p - \tilde{p} - \hat{k} + \hat{r}; \hat{r}, \check{k}) \overline{G}_{1,\tilde{m}}(\tilde{p} + q + \hat{k} - \hat{r}; \hat{k}, \check{r}) G_{1,n}(q; r) G_{2,m}(p; k), \quad (\text{B.33})
\end{aligned}$$

where in the last step we made a change of variables $\tilde{p} \rightarrow \tilde{p} + q + \hat{k} - \hat{r}$. This gives the first term on the r.h.s. of (B.27).

Let us now consider the contribution of the second term in the bracket on the r.h.s. of formula (B.29):

$$\begin{aligned}
& \langle \Omega, B_{\tilde{n}}(G_{2,\tilde{n}})(\check{H}_{1,\sigma}^c)^* B_{\tilde{m}}(G_{1,\tilde{m}}) B_n^*(G_{1,n}) \check{H}_{1,\sigma}^c B_m^*(G_{2,m}) \Omega \rangle_2 \\
&:= \int d^3 w \int d^3 \tilde{q} d^3 \tilde{p} d^3 q d^3 p \int d^{3\tilde{n}} \tilde{r} d^{3\tilde{m}} \tilde{k} d^{3n} r d^{3m} k \check{v}_{\alpha}^{\sigma}(\tilde{p} - \tilde{k} - p + \underline{k} + w) \check{v}_{\alpha}^{\sigma}(w) \\
&\quad \times \overline{G}_{2,\tilde{n}}(q + \tilde{r} - \underline{r}; \tilde{r}) \overline{G}_{1,\tilde{m}}(\tilde{p}; \tilde{k}) G_{1,n}(q; r) G_{2,m}(p; k) \\
&\quad \times \langle \Omega, a(\tilde{r})^{\tilde{n}} a(\tilde{p} - \tilde{k} - p + \underline{k} + w) a(\tilde{k})^{\tilde{m}} a^*(r)^n a^*(w) a^*(k)^m \Omega \rangle \\
&= \int d^3 w \int d^3 \tilde{p} d^3 q d^3 p \int d^{3\tilde{n}} \tilde{r} d^{3\tilde{m}} \tilde{k} d^{3n} r d^{3m} k \check{v}_{\alpha}^{\sigma}(\tilde{p} - \tilde{k} - p + \underline{k} + w) \check{v}_{\alpha}^{\sigma}(w) \\
&\quad \times \overline{G}_{2,\tilde{n}}(q + \tilde{r} - \underline{r}; \tilde{r}) \overline{G}_{1,\tilde{m}}(\tilde{p}; \tilde{k}) G_{1,n}(q; r) G_{2,m}(p; k) \delta(\tilde{p} - \tilde{k} - p + \underline{k}) \\
&\quad \times \langle \Omega, a(\tilde{r})^{\tilde{n}} a(\tilde{k})^{\tilde{m}} a^*(r)^n a^*(k)^m \Omega \rangle \\
&= \|\check{v}_{\alpha}^{\sigma}\|_2^2 \sum_{\rho \in \mathcal{S}_{m+n}} \int d^3 q d^3 p \int d^{3n} r d^{3m} k G_{1,n}(q; r) G_{2,m}(p; k) \\
&\quad \times \overline{G}_{2,\tilde{n}}(q + \tilde{k} - \tilde{r}; \hat{r}, \check{k}) \overline{G}_{1,\tilde{m}}(p - \tilde{k} + \tilde{r}; \hat{k}, \check{r}), \quad (\text{B.34})
\end{aligned}$$

where in the first step we integrated over \tilde{u}, \tilde{w}, u and in the last step we made use again of Lemma B.2. This gives the second term on the r.h.s. of (B.27) and concludes the proof. \square

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